Introduction to dependency logic

Tractability frontier in team semantics

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January 6, 2015

- A signature σ is a set of symbols.
- A model $\mathcal{M}$ over σ is a set M and an interpretation of the signature's symbols.
- A set of variables V.
- If R is a symbol of σ with arity k and $x_1, \ldots, x_k \in V$, then $R(x_1, \ldots, x_k)$ is a term.
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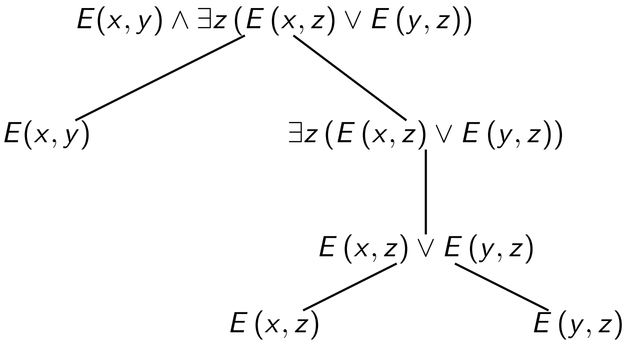
Remark

In all our talk, every model $\mathcal{M}$ will be finite (i.e., $|M| \in \mathbb{N}$).

Let x, y be some variables, ϕ and ψ some first order formulas and s an assignment. A formula is satisfiable by $\mathcal{M}$ and s if:

- $(\mathcal{M}, s) \models R(x_1, \ldots, x_k)$ iff $(s(x_1), \ldots, s(x_k)) \in R^{\mathcal{M}}$
- $(\mathcal{M}, s) \models \phi \wedge \psi$ iff $(\mathcal{M}, s) \models \phi$ and $(\mathcal{M}, s) \models \psi$
- $(\mathcal{M}, s) \models \phi \vee \psi$ iff $(\mathcal{M}, s) \models \phi$ or $(\mathcal{M}, s) \models \psi$
- $(\mathcal{M}, s) \models \exists x \phi$ iff there exists at least one element $a \in M$ such that if we write $s[a/x]$ the prolongation of s to x such that $s[a/x](x) = a$, $\mathcal{M} \models_{X[a/x]} \phi$
- $(\mathcal{M}, s) \models \forall x \phi$ iff for any $a \in M$, $\mathcal{M} \models_{X[a/x]} \phi$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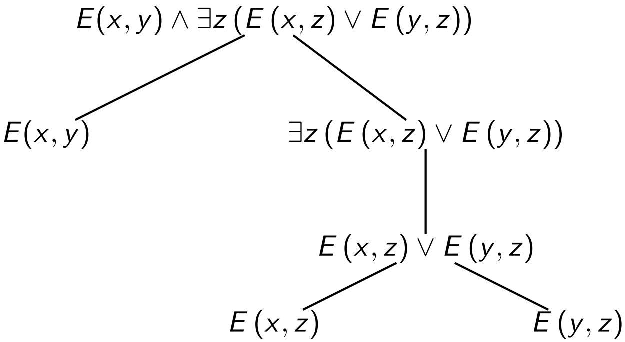
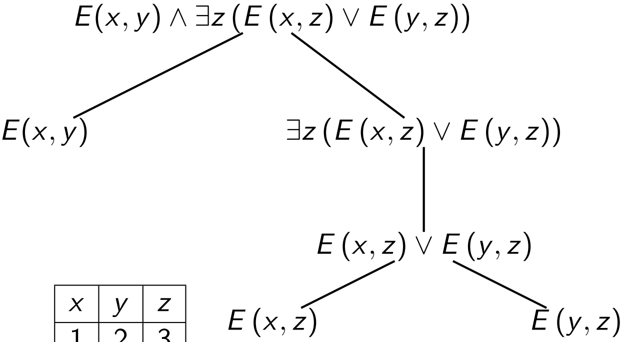
$E\,(x,z) \vee E\,(y,z)$
$E\,(x,z)$
$E\,(y,z)$

x	y
1	2



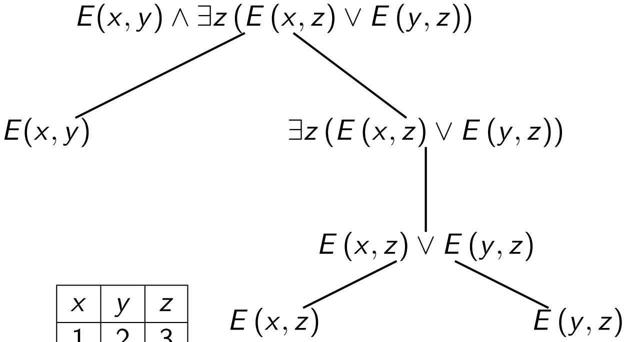


x	y
1	2

x	y	z
1	2	3

x	y	z
1	2	3

$$E^{\mathcal{M}} = \{ (1,2), (2,3), (3,4), (4,1), (1,3) \}$$



x	y
1	2

x	y	z
1	2	3

x	y	z
1	2	3

Definition

Let V be a set of variables and $\mathcal{M}$ a model. *A team* X over V and $\mathcal{M}$ is a set of assignments $X = \{ s : V \to M \}$

Let x, y be some variables, ϕ and ψ some first order formulas and X a team. $\mathcal{M}$ and X satisfy a formula if:

- $\mathcal{M} \models_X R(x_1, \dots, x_k)$ iff for all $s \in X$, $(s(x_1), \dots, s(x_k)) \in R^{\mathcal{M}}$
- $\mathcal{M} \models_X \phi \wedge \psi$ iff $\mathcal{M} \models_X \phi$ and $\mathcal{M} \models_X \psi$
- $\mathcal{M} \models_X \phi \vee \psi$ iff $X = Y \cup Z$ and $\mathcal{M} \models_Y \phi$ and $\mathcal{M} \models_Z \psi$
- $\mathcal{M} \models_X \exists x \phi$ iff there exists some function $F : X \to \mathcal{P}(M)$ such that $\mathcal{M} \models_{X[F/x]} \phi$
- $\mathcal{M} \models_X \forall x \phi$ iff $\mathcal{M} \models_{X[M/x]} \phi$

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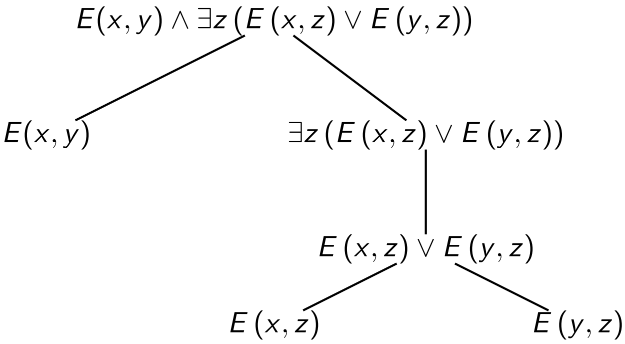
Theorem

If ϕ is a first order formula and X is a team, then,

$$\mathcal{M} \models_X \phi \text{ if and only if for all } s \in X,\ \mathcal{M} \models_s \phi$$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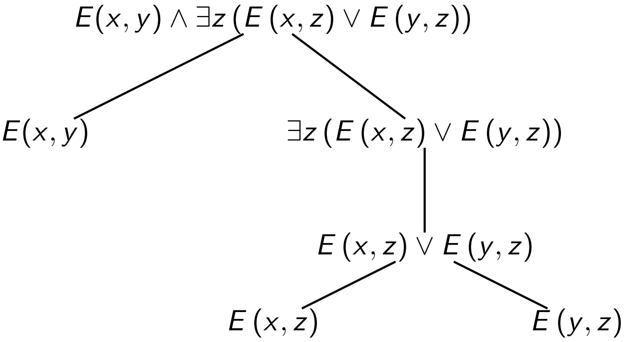
X

x	y
1	2
1	3
2	3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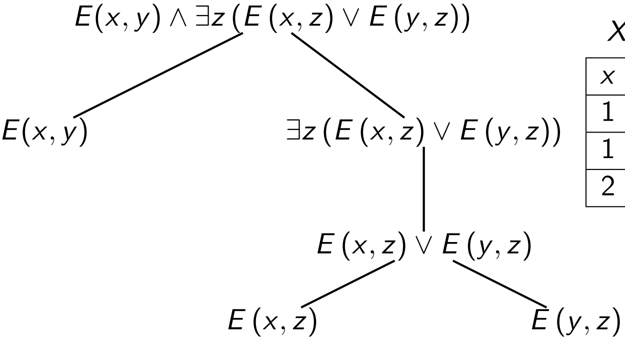

$X[F/z]$

x	y	z
1	2	1
1	3	4
2	3	1

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X

x	y
1	2
1	3
2	3

$X[F/z]$

x	y	z
1	2	1
1	3	4
2	3	1

x	y	z
1	3	4
2	3	1

Y

x	y	z
1	2	1

Z

question

In the formula

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \phi(x_1, x_2, y_1, y_2)$$

y_2 depends on x_1, y_1 and x_2. How can we express that y_1 depends on x_1 and y_2 on x_2 only ?

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Henkin partially ordered quantifiers are formulas of the form:

$$\begin{pmatrix} \forall x_1 & \exists y_1 \\ \forall x_2 & \exists y_2 \end{pmatrix} \phi(x_1, x_2, y_1, y_2)$$

question

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Hintikka and Sandu express the independence between variables by quantifiers of the form $\exists y / \forall x$ in the so call Independence Friendly Logic:

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 / \forall x_1 \phi(x_1, x_2, y_1, y_2)$$

question

In the formula

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \phi(x_1, x_2, y_1, y_2)$$

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Väänänen introduced the indepence logic:

$$\forall x_1 \forall x_2 \exists y_1 \exists y_2 \left(\phi\left(x_1, x_2, y_1, y_2\right) \wedge y_2 \perp x_1\right)$$

Definition

Let $\mathcal{M}$ be a model, $\bar{x}, \bar{y}$ some variables, X a team.

- $\mathcal{M} \models_X =\!(\bar{x}, y)$ iff for all $s, s' \in X$, if $s(\bar{x}) = s'(\bar{x})$, then $s(y) = s'(y)$.
  The first order logic augmented by this atom $=(.,.)$ is called dependence logic and written $\mathrm{FO}(\mathcal{D})$.

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- $\mathcal{M} \models_X \bar{x} \perp \bar{y}$ iff $\forall s, s' \in X$, there exists $s'' \in X$ such that $s(\bar{x}) = s''(\bar{x})$ and $s'(\bar{y}) = s''(\bar{y})$.
  The first order logic augmented by this atom $\perp$ is called independence logic and written $\mathrm{FO}(\perp)$.

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  The first order logic augmented by this atom $\perp$ is called independence logic and written $\mathrm{FO}(\perp)$.
- $\mathcal{M} \models_X \bar{x} \subseteq \bar{y}$ iff $X(\bar{x}) \subseteq X(\bar{y})$. Or for all $s \in X$, there exists $s' \in X$ such that $s(\bar{x}) = s(\bar{y})$.
  The first order logic augmented by this atom $\perp$ is called inclusion logic and written $\mathrm{FO}(\subseteq)$.

x	y	z
1	0	0
1	0	1
0	1	0
0	1	1

Theorem

Let $\mathcal{M} = \{0, 1\}$ a model on an empty signature, X the above team. Then

$$\mathcal{M} \models_X =(x, y)$$

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1	0	0
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0	1	0
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Theorem

Let $\mathcal{M} = \{0, 1\}$ a model on an empty signature, X the above team. Then

$$\mathcal{M} \not\models_X = (x, z)$$

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0	1	0
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$$\mathcal{M} \not\models_X x \perp y$$

$(1, 0)$ *and* $(0, 1)$ *are in* $X(x, y)$ *but not* $(1, 1)$

x	y	z
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1	0	1
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Theorem

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Theorem (Jarmo Kontinen 2010)

For every $3-\mathrm{SAT}$ instance θ, there is a model $\mathcal{M}$ and a team X such that

$$\mathcal{M} \models_X = (x,y) \vee = (z,v) \vee = (z,v) \text{ iff } \theta(p_1,\ldots,p_m) \text{ is satisfiable.}$$

$\theta = \bigwedge_{i=1}^{k} C_i$ where C_i are 3-clause, disjunction of three variables p_i or $\neg p_i$. The sub-team associated to a clause $C_i = p_{i_1} \vee \neg p_{i_2} \vee p_{i_3}$ is:

clause	var	parity	position
z	x	y	v
i	p_{i_1}	1	0
i	p_{i_2}	0	1
i	p_{i_3}	1	2

Definition

The model checking in data complexity(i.e. for a fixed formula ϕ) is the following problem:

- **Input**: a model $\mathcal{M}$ and a team X.
- **Question**: Is $\mathcal{M} \models_X \phi$?

Corollary (Jarmo Kontinen 2010)

The model checking in data complexity for formula in $\mathrm{FO}(\mathcal{D})$ *is* NP*-complete.*

Definition

Let L be a logic and C a complexity class. We say that the logic L characterizes C if for any signature σ, any class of models $\mathcal{K}$ on σ, the two following conditions are equivalent:

- $\mathcal{K}$ is definable in L i.e. there exists a formula $\phi \in L$ such that for each model $\mathcal{M} \in M^k$, $\mathcal{M} \in \mathcal{K}$ iff $\mathcal{M} \models \phi$.
- $\mathcal{K} \in C$ i.e., the problem : given a model $\mathcal{M} \in M^r$, is $\mathcal{M} \in K$? is in C.

Theorem (Fagin 1973)

Existential second order logic (written ESO *or* Σ^1_1*) characterize* NP*.*

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Theorem (Väänänen 2007)

$\mathrm{FO}(\mathcal{D})$ *characterize* NP.

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Theorem (Galliani 2013)

$\mathrm{FO}(\subseteq)$ *characterize* P *on ordered structures.*

Theorem

The model-checking (with fixed formula) for formula of the form

$$\phi \equiv (x \perp y) \vee (x \perp y) \vee (x \perp y) \vee x \neq y$$

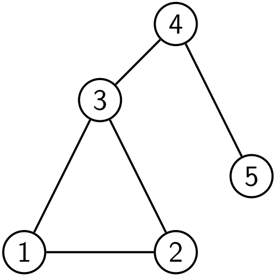
is NP-complete

Let G be a graph which vertices are $\{ v_1, \ldots, v_n \}$, $\mathcal{M} = \{V_G\}$ and X be the team
$\{ (v, v) \mid v \in V_G \} \cup \{ (v_1, v_2), (v_2, v_1) \mid (v_1, v_2) \in E_G \}$:

x	y
1	1
2	2
3	3
4	4
5	5

$\cup$

x	y
1	2
2	1
2	3
3	2
3	1
1	3
3	4
4	3
4	5
5	4



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$\cup$

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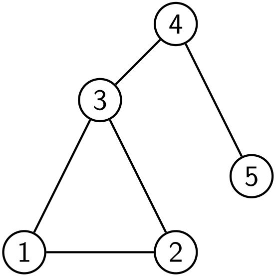

Proposition

Then G has a 3-clique cover if and only if
$\mathcal{M} \models_X (x \perp y) \vee (x \perp y) \vee (x \perp y) \vee x \neq y.$

x	y
1	1
2	2
3	3
4	4
5	5

$\cup$

x	y
1	2
2	1
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3	2
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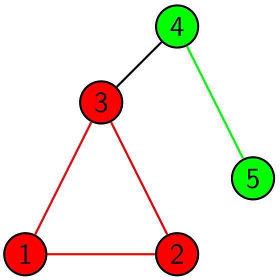

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$\mathcal{M} \models_X (x \perp y) \vee (x \perp y) \vee (x \perp y) \vee x \neq y.$

x	y
1	1
2	2
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4	4
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$\cup$

x	y
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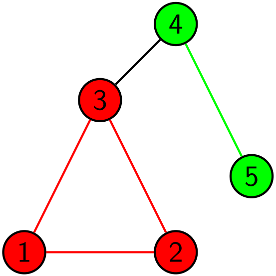

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x	y
1	1
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$\cup$

x	y
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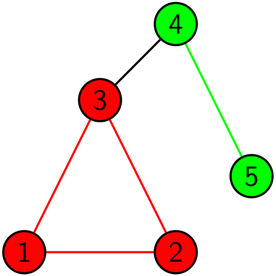
4	5
5	4



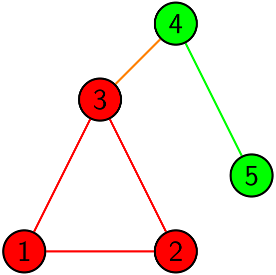
Proposition

If G has a 3-clique cover then
$\mathcal{M} \models_X (x \perp y) \vee (x \perp y) \vee (x \perp y) \vee x \neq y.$

x	y
1	1
2	2
3	3
4	4
5	5

$\cup$

x	y
1	2
2	1
2	3
3	2
3	1
1	3
3	4
4	3
4	5
5	4



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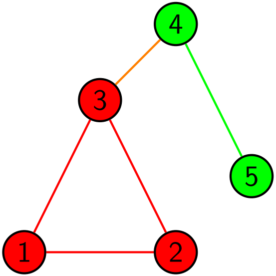

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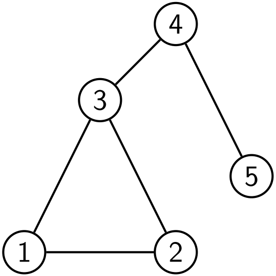

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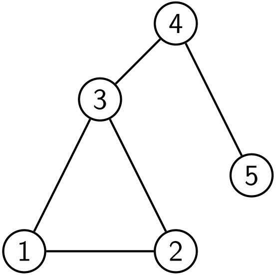

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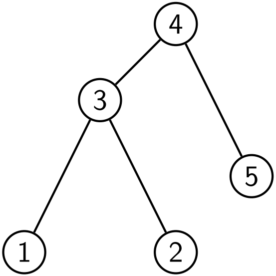

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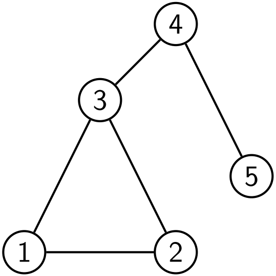

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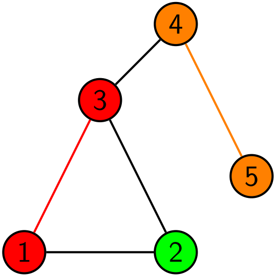

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