Determinant versus Permanent: salvation via generalization?

The algebraic complexity of the Fermionant and the Immanant

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The size of the entry is the number of variables. The size of an arithmetic circuit is the number of operational gates.

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Definition
A family \( F = (f_n) \) of polynomials is in \( VP \) if there exists a family of circuits \( C_n \) of polynomial size such that for any \( n \)

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f_n \text{ is computed by } C_n
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Definition
A family \( F = (f_n) \) is in \( VNP \) if there is a family \( G = (g_n) \) in \( VP \) such that

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f_n(\bar{x}) = \sum_{\bar{\epsilon} \in \{0,1\}^n} g_n(\bar{\epsilon}, \bar{x})
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Definition

Let $S_n$ be the symmetric group on $n$ elements and if $\pi \in S_n$, $c(\pi)$ its number of cycles. The determinant is

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det_n(\bar{x}) = (-1)^n \sum_{\pi \in S_n} (-1)^{c(\pi)} \prod_{i=1}^{n} x_{i\pi(i)}$$

Theorem (Valiant 79)

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**Definition**

Let $S_n$ be the symmetric group on $n$ elements and if $\pi \in S_n$, $c(\pi)$ its number of cycles. The permanent is

$$\text{per}_n(\bar{x}) = \sum_{\pi \in S_n} \prod_{i=1}^{n} x_{i\pi(i)}$$

**Theorem (Valiant 79)**

The family $\text{per} = (\text{per}_n)_{n \in \mathbb{N}}$ is VNP-complete.
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Conjecture (Valiant hypothesis)

$\text{VP} \neq \text{VNP}$

Theorem (Bürgisser 2000)

Under Generalized Riemann Hypothesis,

$\text{VP} = \text{VNP} \Rightarrow \text{P/poly} = \text{NP/poly}$
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The main approaches of Valiant Hypothesis:

- Geometric Complexity Theory (GCT)
- Lower bounds
- The study of complexity classes (Characterization, complete polynomials)
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Definition (informal)

A generalization of the determinant and the permanent is a series of family \( \mathbf{F}^k = (f_n)^k \) indexed by some \( k \) such that

- For certain \( k \), \( \mathbf{F}^k \) are in \( \text{VP} \)
- For others \( k \), \( \mathbf{F}^k \) are \( \text{VNP-complete} \).
If $\pi$ is a permutation, $c(\pi)$ is its number of cycles.

$$\text{Ferm}_n^k A = (-1)^n \sum_{\pi \in S_n} (-k)^{c(\pi)} \prod_{i=1}^{n} A_{i,\pi(i)}$$

Let $\text{Ferm}_n^k$ the family of $(\text{Ferm}_n^k)$

- If $k = 1$, then $\text{Ferm}_n^1(\bar{x}) = \det(\bar{x})$
- If $k = -1$, then $\text{Ferm}_n^{-1}(\bar{x}) = \text{per}(\bar{x})$
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Theorem (1)

- $\text{Ferm}^0 = 0$.
- $\text{Ferm}^1$ is in VP
- for $k \in \mathbb{Q}$ different from 0, 1 $\text{Ferm}^k$ is VNP-complete.

Corollary

$\text{Ferm}^k$ is $\#P$-complete for any rational $k \notin \{0, 1\}$. 

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**Theorem (1)**

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Lemma (Idea of the demonstration)

Let $X = (x_{i,j})_{i,j \in [n]}$ be the matrix of variables. For any $l \in \mathbb{N}$ there exists a transformation $P_l$ of $X$ such that

$$\text{Ferm}^k(P_l(X)) = \sum_{\pi \in S_n} (-k)^{c(\pi) \times l} \prod_{i=1}^{n} x_{i,\pi(i)}$$
Definition

A young diagram is a collection of boxes in left adjusted row with decreasing row length.

\[
\begin{array}{c}
\text{Young diagrams} \\
[4, 4] & [4, 2] \\
\end{array}
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Definition

Let \( \chi_Y \) be an irreducible character of \( S_n \). Then

\[
im_{\chi}(\bar{x}) = \sum_{\pi \in S_n} \chi_Y(\pi) \prod_{i=1}^{n} x_{i,\pi(i)}
\]
Exemple
If $Y$ is a single row, then $\text{im}_Y(\bar{x}) = \text{per}(\bar{x})$.

Exemple
If $Z$ is a single column, then $\text{im}_Z(\bar{x}) = \text{det}(\bar{x})$. 
Theorem (Bürgisser 2000)

If \((Y_n)\) is a family of Young diagrams with only a constant number of boxes at the right of the first column, then

\[(im_{Y_n})\] is in VP
Theorem (Brylinski 2003)

Let $Y_n$ be Young diagrams such that the maximal difference between the size of two consecutive rows is $\Omega(n)$, then

$(\text{im} Y_n)$ is VNP-complete
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Theorem (2)

Let \([n, n]\) be the Young diagram with two columns, each with \(n\) boxes. Then

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Theorem (3)

If \((Y_n)\) has a polynomial number of boxes at the right of the first column and a constant number of columns, then

\((\text{im}_{Y_n})\) is VNP-complete
Proposition

For any integers $k$, $n$, if we write $\Lambda_{k}^{n}$ the set of every Young diagrams with $n$ boxes and at most $k$ columns, then there exists some rational constants $d_{Y}^{k}$ such that for any matrix $A$,

$$\text{Ferm}_{n}^{k}(A) = \sum_{Y \in \Lambda_{k}^{n}} d_{Y}^{k} \text{im}_{Y}(A)$$
Theorem (Conclusion)

Let \((Y_n)\) be a family of Young diagrams with a constant number of columns such that \(|Y_n| = \Omega(n)\). Then

- If the number of boxes at the right of the first column is constant \(c\), then \((\text{im}_{Y_n})\) is in \(\text{VP}\).
- If the number of boxes at the right of the first column is logarithmic, then \((\text{im}_{Y_n})\) is not \(\text{VNP}\)-complete.
- If the number of boxes at the right of the first column is polynomial, \((\text{im}_{Y_n})\) is \(\text{VNP}\)-complete.

Perspectives

- Studying the class of polynomials computed by sub-exponentiel circuits.
- Finding a \(\text{VP}\)-complete family!
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Perspectives

- Studying the class of polynomials computed by sub-exponentiel circuits.
- Finding a VP-complete family!
Thank you!