# Determinant versus Permanent: salvation via generalization?

## The algebraic complexity of the Fermionant and the Immanant

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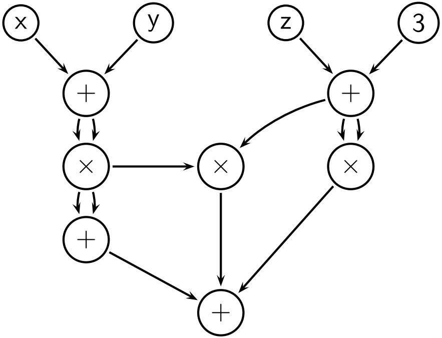
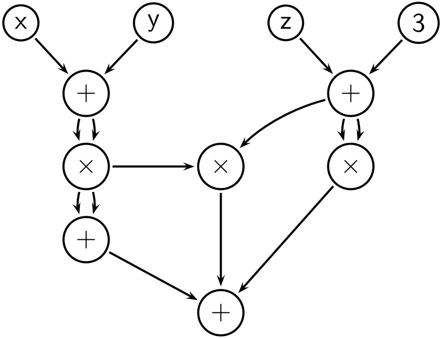



$$f(x,y) = (x+y)^2(z+3) + 2(x+y)^2 + (z+3)^2$$

**Definition**

The size of the entry is the number of variables. The *size* of an arithmetic circuit is the number of operational gates.

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A family $\mathbf{F} = (f_n)$ of polynomials is in $\mathrm{VP}$ if there exists a family of circuits $C_n$ of polynomial size such that for any $n$

$f_n$ is computed by $C_n$

### Definition

A family $\mathbf{F} = (f_n)$ is in $\mathrm{VNP}$ if there is a family $\mathbf{G} = (g_n)$ in $\mathrm{VP}$ such that

$$f_n(\bar{x}) = \sum_{\bar{\epsilon} \in \{0,1\}^n} g_n(\bar{\epsilon}, \bar{x})$$

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**Definition**

Let $S_n$ be the symmetric group on $n$ elements and if $\pi \in S_n$, $c(\pi)$ its number of cycles. The determinant is

$$\det_n(\bar{x}) = (-1)^n \sum_{\pi \in S_n} (-1)^{c(\pi)} \prod_{i=1}^{n} x_{i\pi(i)}$$

**Theorem (Valiant 79)**

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$$\mathsf{per}_n(\bar{x}) = \sum_{\pi \in S_n} \prod_{i=1}^{n} x_{i\pi(i)}$$

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**Conjecture (Valiant hypothesis)**

$$\mathrm{VP} \neq \mathrm{VNP}$$

**Theorem (Bürgisser 2000)**

*Under Generalized Riemann Hypothesis,*

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**Definition (informal)**

A generalization of the determinant and the permanent is a series of family $\mathbf{F}^k = (f_n)^k$ indexed by some $k$ such that

- For certain $k$, $\mathbf{F}^k$ are in $\mathrm{VP}$
- For others k, $\mathbf{F}^k$ are $\mathrm{VNP}$-complete.

## Definition

If $\pi$ is a permutation, $c(\pi)$ is its number of cycles.

$$\text{Ferm}_n^k A = (-1)^n \sum_{\pi \in S_n} (-k)^{c(\pi)} \prod_{i=1}^{n} A_{i,\pi(i)}$$

Let $\mathbf{Ferm}^k$ the family of $(\text{Ferm}_n^k)$

- If $k = 1$, then $\text{Ferm}^1(\bar{x}) = \det(\bar{x})$
- If $k = -1$, then $\text{Ferm}^{-1}(\bar{x}) = \text{per}(\bar{x})$

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**Theorem (1)**

- *Ferm*$^0 = 0$.
- *Ferm*$^1$ *is in* $\mathrm{VP}$
- *for* $k \in \mathbb{Q}$ *different from* $0, 1$ *Ferm*$^k$ *is* $\mathrm{VNP}$*-complete.*

**Corollary**

*Ferm*$^k$ *is* $\#P$*-complete for any rational* $k \notin \{0, 1\}$.

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**Lemma (Idea of the demonstration)**

*Let $X = (x_{i,j})_{i,j \in [n]}$ be the matrix of variables. For any $l \in \mathbb{N}$ there exists a transformation $P_l$ of $X$ such that*

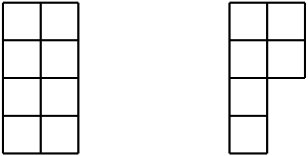
$$\mathit{Ferm}^k(P_l(X)) = \sum_{\pi \in S_n} (-k)^{c(\pi) \times l} \prod_{i=1}^{n} x_{i,\pi(i)}$$

## Definition

A young diagram is a collection of boxes in left adjusted row with decreasing row length.

**Young diagrams**

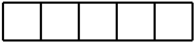
$[4, 4]$ $[4, 2]$



## Definition

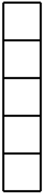
Let $\chi_Y$ be an irreducible character of $S_n$. Then

$$\text{im}_\chi(\bar{x}) = \sum_{\pi \in S_n} \chi_Y(\pi) \prod_{i=1}^{n} x_{i,\pi(i)}$$

$Y =$ 

### Exemple

If $Y$ is a single row, then $\mathrm{im}_Y(\bar{x}) = \mathrm{per}(\bar{x})$.

$Z =$ 

### Exemple

If $Z$ is a single column, then $\mathrm{im}_Z(\bar{x}) = \det(\bar{x})$.

**Theorem (Bürgisser 2000)**

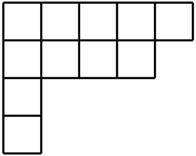
*If $(Y_n)$ is a family of Young diagrams with only a constant number of boxes at the right of the first column, then*

$$(im_{Y_n}) \text{ is in } \mathrm{VP}$$

**Theorem (Brylinski 2003)**

*Let $Y_n$ be Young diagrams such that the maximal difference between the size of two consecutive rows is $\Omega(n)$, then*

$$(\mathit{im}_{Y_n}) \text{ is } \mathrm{VNP}\text{-complete}$$



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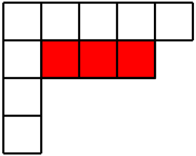
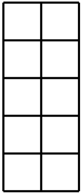
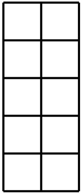


$[n, n] =$ 

**Theorem (2)**

*Let $[n, n]$ be the Young diagram with two columns, each with $n$ boxes. Then*

$$(im_{[n,n]}) \text{ is VNP-complete}$$

$[n, n] =$ 

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**Theorem (3)**

*If $(Y_n)$ has a polynomial number of boxes at the right of the first column and a constant number of columns, then*

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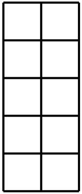
$[n, n] =$ 

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**Proposition**

*For any integers $k, n$, if we write $\Lambda_k^n$ the set of every Young diagrams with $n$ boxes and at most $k$ columns, then there exists some rational constants $d_Y^k$ such that for any matrix $A$,*

$$\mathit{Ferm}_n^k(A) = \sum_{Y \in \Lambda_k^n} d_Y^k \, \mathit{im}_Y(A)$$

## Theorem (Conclusion)

*Let $(Y_n)$ be a family of Young diagrams with a constant number of columns such that $|Y_n| = \Omega(n)$. Then*

- *If the number of boxes at the right of the first column is constant $c$, then $(im_{Y_n})$ is in $\mathrm{VP}$.*
- *If the number of boxes at the right of the first column is logarithmic, then $(im_{Y_n})$ is not $\mathrm{VNP}$-complete*
- *If the number of boxes at the right of the first column is polynomial, $(im_{Y_n})$ is $\mathrm{VNP}$-complete.*

## Perspectives

- Studying the class of polynomials computed by sub-exponentiel circuits.
- Finding a $\mathrm{VP}$-complete family !

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Thank you !