Introduction to forcing axioms and the cardinality of the continuum

Giorgio VENTURI

SNS

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We will work in ZFC, first order formalization of set theory. Here there are just and only sets.

$V$ stands for the universal class, defined with the formula $\phi(x) = \{x : x = x\}$. It can be seen in a cumulative way:

\[
\begin{align*}
V_0 &= \emptyset \\
V_{\alpha + 1} &= P(V_\alpha) \\
V_\lambda &= \bigcup_{\mu < \lambda} V_\mu \quad \text{per} \ \lambda \ \text{limite}.
\end{align*}
\]

And finally $V = \bigcup_{\alpha \in \text{Ord}} V_\alpha$. 

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Models and independence

A model of set theory is a set $M$ with a relation $E \subseteq M \times M$, such that, if $\phi(x)$ is a formula written in the language of set theory, then

- $\text{ZFC} \vdash \phi(x) \Rightarrow \exists x \in M \models \phi(x)$,
- $\exists x \in M \models \phi(x) \Rightarrow \text{ZFC} \nvdash \neg \phi(x)$,
- $x \in y \iff \exists x, y \in M \left( x E y \right)$.

Note that $V$ is not a model, since it is not a set.

Definizione

A sentence $\phi$ written in the language of set theory is independent from ZFC if there are two different models $M$ e $M'$ such that

$$M \models \text{ZFC} + \phi \quad \text{e} \quad M' \models \text{ZFC} + \neg \phi.$$
How to prove independence

To show the independence of a sentence \( \phi \) we use two different methods:

1. the method of inner models, and
2. the method of forcing (outer model).

The two methods are complementary, since the first one shows the coherence of a sentence, i.e. it builds a model \( M \) such that \( M \models \phi \); while the second one proves its consistency, showing that there is an other model \( N \) such that \( N \models \neg \phi \).
Thanks to the first method, given a model $M$ and a sentence $\phi$ independent from $M$, we are able to build a new model $M'$ such that $M' \subseteq M$ and $M' \models \phi$. What really happens is that we narrow the domain of $M$, keeping the same the relation of logical consequence as the one defined for the $M$. In this way, in the new model, we eliminate the counterexample that make false the sentence $\phi$.

This method was invented by Gödel, in 1938. He built up the minimal inner model: $L$, said the constructible universe, since in $L$ there are just the sets that have a predicative definition.
What is forcing

The method of forcing, given a transitive and countable model $M$ (ground model) and a sentence $\phi$ independent from $M$, allows us to find a new model $M[G]$ (generic extension) such that $M \subseteq M[G]$ and $M[G] \models \neg \phi$.

There is a crucial difference between forcing and the method of inner models; in the generic extension the relation of logical consequence is not the same of the ground model. Indeed it depends on some conditions $p$ that belong to the ground model. This is one of the reasons why truth in the generic extension depends on a relation defined inside the ground model, namely the relation of forcing that is written $\Vdash$ and has the following property:

$$p \Vdash \phi \iff M[G] \models \phi.$$
Partial orders

The main goal of forcing is to extend a model, showing that in the generic extension there are new objects. The conditions $p$, that determine the truth of the sentences in the generic extension, are partial descriptions (very often, finite pieces of informations) of a new object $G$, such that $G \in M[G]$, but $G \notin M$.

Making a set out of the conditions, we can define a partial order $P = (\{p : \text{è una condizione}\}, \leq)$, (that is called a notion of forcing, or a forcing) where $p \leq q$ ($p$ extends $q$) iff $p$ has more informations than $q$. Note that the order is reverse than expected; the idea behind is that the extension of a condition $p$ has less freedom in imposing new property to $G$, than $p$ has.
Generic filters I

Since we want to construct a coherent object, we need compatible conditions; the idea is then to refine the set $P$ and so deal with a subset of $P$, where each conditions is compatible with the others. Moreover, if we look at a condition $p \in P$ as a set of informations to impose to the new object, we want that a subset of the information given by $p$ were still in our refinement of $P$. This set is then a filter, say $G$, since

- $\forall p, q \in G, \exists r \in G$ s.t. $r \leq q$ and $r \leq p$, hence $r$ witnesses that $p$ and $q$ are compatible,
- $\forall p \in G$, if $p \leq q$, then $q \in G$. 

Let's consider the following properties of $G$:

1. $\forall p, q \in G, \exists r \in G$ s.t. $r \leq q$ and $r \leq p$, hence $r$ witnesses that $p$ and $q$ are compatible,
2. $\forall p \in G$, if $p \leq q$, then $q \in G$. 

These properties define a generic filter.
We say that a set $D \subseteq P$ is dense if $\forall p \in P \exists q \in D (q \leq p)$. If $P$ is a notion of forcing, a dense set $D$ is a set of conditions with some properties that the new object $G$, can’t avoid, because given a condition $p$, sooner or later there will be an extension of $p$ with that property.

We say that a filter $G$ is $M$-generic if for every dense $D \in M$, $G \cap D \neq \emptyset$.

**Teorema**

*(Cohen)* Given a countable transitive model $M$ and a separative poset $P$ there is a $M$-generic filter $G$, such that $G \notin M$. 
The method of forcing was invented by Cohen, in 1963. By means of forcing Cohen could show the independence of the Continuum Hypothesis (CH): $2^{\aleph_0} = \aleph_1$.

**Theorem**

*(Gödel, ‘38)* $\text{Con} (\text{ZFC}) \Rightarrow \text{Con} (\text{ZFC} + \text{CH}).$

**Theorem**

*(Cohen, ‘63)* $\text{Con} (\text{ZFC}) \Rightarrow \text{Con} (\text{ZFC} + \neg \text{CH}).$
Definizione

**Forcing Axiom:** $\text{FA}(\Gamma, \kappa)$ holds if, for every poset $P$ with the property $\Gamma$, given $\mathcal{D} = \{D_\alpha \subseteq P : \alpha \leq \kappa\}$ a family of $\kappa$ dense subsets of $P$, there is a filter $G \subseteq P$ that intersects every $D_\alpha$.

Note that we are not asking $G$ to intersect every dense of $P$, indeed maybe $G$ is not $M$-generic for every $M$, countable transitive model; hence it may happen that can’t build a generic extension out of $G$. 

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There is an equivalent topological definition of the forcing axioms.

**Definizione**

**Forcing Axiom:** \( FA(\mathcal{A}, \kappa) \) holds if for a given class \( \mathcal{A} \) of topological spaces, if \( X \in \mathcal{A} \) and for every family \( \mathcal{F} \) of size at most \( \kappa \) of open dense subsets of \( X \), we have that \( \bigcap \mathcal{F} \neq \emptyset \).

This definition shows how Forcing Axioms are a generalization of Baire category theorem.
Teorema

Baire category theorem: given a family $\mathcal{F}$ of open dense subsets of $\mathbb{R}$ such that $|\mathcal{F}| \leq \aleph_0$, then $\bigcap \mathcal{F} \neq \emptyset$.

This theorem holds in every separable completely metrizable topological space, where you can show that there is a countable base of open sets, hence every intersection of uncountably many open sets is already an intersection of countable many base open sets.

If we call $\Gamma_{CS}$ the class of separable completely metrizable topological space, we have that the Baire category theorem is $FA(\Gamma_{CS}, \aleph_0)$ and that is a theorem of ZFC.
Historically the first Forcing Axiom to be defined was Martin’s Axiom. It deals with c.c.c. posets, where the antichains have size at most countable.

**Definizione**

**Martin’s Axiom (MA):** $\text{MA}(\kappa)$ holds if, given $P$ a c.c.c. poset and a family $\mathcal{D}$ of dense subsets of $P$, such that $|\mathcal{D}| \leq \kappa$, then there is a filter $G \subseteq P$ such that $G \cap D \neq \emptyset$, for every $D \in \mathcal{D}$.

$\text{MA}$ holds if $\text{MA}(\kappa)$ holds for every $\kappa < 2^{\aleph_0}$.

Hence we have that $\text{MA}(\kappa) = \text{FA}(\Gamma_{\text{c.c.c.}}, \kappa)$. 
The topological version of c.c.c. posets are the c.c.c. topological spaces.

**Definizione**

A topological space $X$ is c.c.c. if there is no family of pairwise disjoint, non empty, open sets, of size grater that $\aleph_0$.

It’s easy to see that if we have a countable open base for the topology, we can’t have an uncountable family of pairwise disjoint, non empty, open sets. Hence $FA(\Gamma_{cs}, \aleph_0) \Rightarrow MA(\aleph_0)$ and $MA(\aleph_0)$ is a theorem of ZFC.
MA e CH

We have just shown that, under CH, MA is already a theorem of ZFC (because, under CH, $\text{MA} = \text{MA}(\aleph_0)$). Nevertheless $\text{MA}(\aleph_1)$ is stronger than ZFC. Indeed:

**Teorema**

$\text{MA}(\aleph_1)$ implies the failure of CH.

**Proof.**

Assume that CH holds, we then can exhibit an enumeration of the reals is order type $\omega_1$: $\mathbb{R} = \{ r_{\alpha} : \alpha \in \omega_1 \}$. But now define for every $\alpha \in \omega_1$, $D_\alpha = \{ h \in \mathbb{R} : h \neq r_{\alpha} \}$. $D_\alpha$ is dense open in $\mathbb{R}$ for every $\alpha \in \omega_1$. Hence every real that belongs to the intersection of all the $D_\alpha$ is a real that does not belong to the previous enumeration. Contradiction.
The classes of posets that give rise of useful Forcing Axioms are the Proper one and the Stationary Set Preserving (SSP) one. We just need to know that $\Gamma_{c.c.c.} \subseteq \Gamma_{\text{proper}} \subseteq \Gamma_{\text{SSP}}$.

The corresponding forcing axioms are:

**Definizione**

*Proper Forcing Axiom (PFA)*: $\text{FA}(\Gamma_{\text{proper}}, \aleph_1)$.

**Definizione**

*Martin’s Maximum (MM)*: $\text{FA}(\Gamma_{\text{SSP}}, \aleph_1)$.

So we have that $\text{MM} \Rightarrow \text{PFA} \Rightarrow \text{MA}(\aleph_1)$. 
Since MM and PFA imply $MA(\aleph_1)$, then they negate CH. Moreover, even if it’s possible to show that MA is independent from CH, they decide the cardinality of the continuum.

**Theorema**

*Assuming MM, we have that for every regular cardinal $\kappa \geq \aleph_2$, $\kappa^{\aleph_1} = \kappa$.***

**Corollario**

*MM implies $2^{\aleph_0} = \aleph_2$.***

**Proof.**

Thanks to the theorem $2^{\aleph_0} \leq 2^{\aleph_1} \leq \aleph_2^{\aleph_1} = \aleph_2$. But MM implies $MA(\aleph_1)$, so $\aleph_1 < 2^{\aleph_0}$; hence $2^{\aleph_0} = \aleph_2$. 

□
**Theorem**  
*Assuming PFA we have $2^\aleph_1 = \aleph_2$.*

**Corollary**  
*PFA implies $2^{\aleph_0} = \aleph_2$.*

**Proof.**  
Thanks to the theorem $2^{\aleph_1} = \aleph_2$. Moreover PFA implies MA($\aleph_1$), so $\aleph_1 \neq 2^{\aleph_0}$; hence $2^{\aleph_0} = \aleph_2$. \qed
Definition

**The Consistency strength** of a theory $T$ is the logical strength of the following sense: “$T$ is consistent”. $T$ has stronger consistency strength than a theory $S$ if we can prove (in the arithmetic) the sentence: “if $T$ is consistent, then $S$ is consistent.”

In general, it is not always possible to compare the consistency strength of two theories, since they do not fall in a linear order. But it is an empirical fact that all the “natural” theories we deal with follow a linear order induced by the consistency strength of large cardinals (the ones, whose existence cannot be prove by means of the axioms of ZFC).
MM and PFA

It is possible to give an upper bound and a lower bound, in terms of large cardinals, of the consistency strength of MM and of PFA.

**Teorema**

*(Foreman, Magidor, Shelah)* If there is a supercompact cardinal, then there is a generic extension in which MM holds; and so also PFA.

**Teorema**

*(Foreman, Magidor, Shelah)* $\text{Con}(\text{ZFC} + \text{MM}) \Rightarrow \text{Con}(\text{ZFC} + \exists$ a proper class of Woodin cardinals).*