An extension of the Lindemann-Weierstrass theorem*

Ana Peón Nieto

23rd June 2008

*Notes on the tutorials given by Daniel Bertrand and Anand Pillay at the MODNET Training Workshop in La-Roche-en-Ardennes, April 2008

Contents

1 When, where, how?

This notes were written after a series of lectures given by Daniel Bertrand and Anand Pillay the MODNET Training Workshop in La-Roche-en-Ardennes (Belgium, April 2008) on their results about an extension of the Lindemann-Weierstrass theorem (paper in preparation under a name isomorphic to “A Lindemann Weierstrass theorem for semi-abelian varieties over function fields”).

We want to thank the lecturers for their help during and after the conference, which has made our task much more enjoyable and especially “understandable”.

Sections 4 and 6 were written by Anand Pillay. We have just added some comments and remarks. The rest of the sections intend to provide the knowledge needed to understand what’s going on.

2 Preliminaries on Lie groups and algebras. The exponential map.

Definition 2.1. *A Lie group G is a differential variety with a group structure such that the map:*

$$\begin{aligned} G \times G &\rightarrow G \\ (x,y) &\mapsto xy^{-1} \end{aligned}$$

is $\mathcal{C}^\infty$. This is equivalent to saying that the group structure is made up of differentiable maps.

Definition 2.2. *A Lie algebra over the field k is a k-vector $\mathfrak{g}$ space admiting the so-called Lie bracket $[\ \cdot\ ,\ \cdot\ ]$, a bilinear map:*

$$\begin{aligned} [\ \cdot\ ,\ \cdot\ ] : \mathfrak{g} \times \mathfrak{g} &\rightarrow \mathfrak{g} \\ (x,y) &\mapsto [x,y] \end{aligned}$$

satisfying:

i) $[x,x] = 0$

ii) $[\ \cdot\ ,\ \cdot\ ]$ *satisfies the Jacobi identity, that is to say:*

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$$

Remark 2.3. *From (i) and bilinearity we deduce $[y,x] = -[x,y]$.*

Example 2.4. *An associative k-algebra A is a Lie algebra with the commutator as Lie brackets $[x,y] = xy - yx$(here the associativity is needed for the Jacobi identity).*

Example 2.5. *Lets take a differentiable manifolf M. We may describe its tangent bundle as the set of vector fields on M. The composition of vector fields is not a vector field (since there are second order derivatives involved), but if we take the Lie bracket of two vector fields, then we do obtain a vector field. Indeed, let $X = \sum_i a_i \frac{\partial}{\partial x_i}$, $Y = \sum_i b_i \frac{\partial}{\partial x_i}$ be two vector fields. Then for any $f \in \mathcal{C}^\infty(M)$:*

$$\begin{aligned} XY(f) &= (\sum_i a_i \frac{\partial}{\partial x_i}) \circ (\sum_j b_j \frac{\partial}{\partial x_j})(f) = \sum_i \sum_j a_i \frac{\partial}{\partial x_i}(b_j \frac{\partial}{\partial x_j}(f)) = \\ &= \sum_i \sum_j a_i (\frac{\partial b_j}{\partial x_i} \frac{\partial f}{\partial x_j} + b_j \frac{\partial^2 f}{\partial x_i \partial x_j}) \end{aligned}$$

Similarly:

$$YX(f) = \sum_j \sum_i b_j \frac{\partial}{\partial x_j}(a_i \frac{\partial}{\partial x_i}(f)) = \sum_i \sum_j b_j (\frac{\partial a_i}{\partial x_j}\frac{\partial f}{\partial x_i} + a_i \frac{\partial^2 f}{\partial x_j \partial x_i})$$

So

$$[X, Y] = \sum_{i,j} (a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i}) \frac{\partial}{\partial x_j}$$

is a vector field.

Definition 2.6. *Given a Lie group G we define its Lie algebra LG to be the tangent space of G at id_G*

Example 2.7. *Key example The easiest case of a Lie group is the torus, $\mathbb{G}_m^n(\mathbb{C}) = (\mathbb{C}^\times)^n$. Note that the map taking $(x_1, .., x_n) \in \mathbb{C}^n$ to $(e^{x_1}, .., e^{x_n}) \in \mathbb{C}^n$, is a surjective complex-analytic homomorphism exp from the complex algebraic group $(\mathbb{C}^n, +)$ to the complex algebraic group $((\mathbb{C}^\times)^n, \times)$. The former is the Lie algebra of the latter.*

Fact: this exponential map makes sense for any Lie group and its Lie algebra. In this case it is taken to be as the only analytic function $f : LG(\mathbb{C}) \to G(\mathbb{C})$ such that its differential at id_G $d_{id_G} f : LG(\mathbb{C}) \to LG(\mathbb{C})$ is the identity on $LG(\mathbb{C})$.

To define the exponential map for arbitrary complex groups, we need some extra notions and facts:

- Fact 1: For any manifold M, the tangent space at a point p is the space of derivations at a that point, that is, vectors seen as operators $v_p : C^\infty(M) \to \mathbb{C}$. In particular, the Lie algebra of a group, that is, the tangents space at id_G.

- Fact 2: Given $X \in LG$ denote by $\mathbf{X}$ the vector field (that is, a map $\mathbf{X} : G \to TG$ such that $\mathbf{X}(g) = \mathbf{X}_g \in T_g G$) invariant by the left action associated (that is $\mathbf{X}(1) = X$, $\mathbf{X}(g) = d_1 l_g(X)$, where $l_g \circ \phi(h) = \phi(gh)$.) Consider the differential equation:

$$\begin{aligned} f'(t) &= \mathbf{X}(f(t)) \\ f(0) &= g \end{aligned} \qquad (1)$$

  Then it has a unique solution α_X^g defined over the whole of $\mathbb{C}$ satyisfying $\alpha_X^g(0) = g\alpha_X^1(0)$(this follows by left invariance.) By uniqueness, we must have $\alpha_{sX}^1(0)\alpha_{tX}^1(0) = \alpha_{tX}^{\alpha_{sX}^1(0)}(0) = \alpha_{(t+s)X}(0)$. Furthermore, we have that $\alpha_{tX}^1(s) = \alpha_X(ts)$. By left invariance, the solution satisfying $f(0) = g$ equals $\alpha_g(t) = g\alpha_1(t)$, thus if $g = \alpha_X^1(0)$ we have that $\alpha_X^1(s + t) = \alpha_X^1(t)\alpha_X^1(s)$. In the same way, $\alpha_{tX}^1(s) = \alpha_X^1(ts)$.

- Fact 3: We define $exp_G : LG \to G$ as the application $X \mapsto \alpha_X(0)$.Now have $exp((t+s)X) = \alpha_{(t+s)X}(0) = \alpha_{tX}(0)\alpha_{sX}(0) = exp(tX)exp(sX)$; de même, $exp(0) = \alpha_0^1(0) = 1$

Example 2.8. *Back to the key example On the torus, the exp above is exactly the usual exponential. The same happens with* $GL_n(\mathbb{C}) and Lie(GL_n(\mathbb{C})) = M_n(\mathbb{C})$ *and the usual exponential map* $t \mapsto exp(tX) = \sum_k \frac{(tX)^k}{k!}$

3 Galois theory of linear differential equations. D-Groups and the logarithmic derivative.

The references for this section are:

i) *Manin Kernels (David Marker, Connections between model theory and algebraic and analytic geometry, Quaderni di Matematica, Univ Naples 2000)*

ii) *Algebraic D-groups and differential Galois theory (Anand Pillay, Pacific Journal Mathematics, vol 216 N°2, pages 342-360 October 2004)*

iii) *Differential Galois theory (Michael Singer, Marius Van Der Put, Michael Singer's website)*

Let us start by the classical example: Picard-Vessiot extensions of a differential field. Let k be a differential field, with algebraically closed field of constants $C_k = \{x \in k : \delta x = 0\}$, and consider a linear differential equation:

$$X' = AX$$

where X is a $m \times m$ matrix of unknowns. By a fundamental matrix of this system we mean a $B \in GL_m(k^{diff})$ satisfying the equation.

Definition 3.1. *Let k be a differential field with $C_k = \{x \in k : \delta x = 0\}$ algebraically closed. A differential ring $R \supset k$ is called a Picard-Vessiot extension of K if*

i) R is simple as a δ-ring, that is, there are no δ-ideals (ideals stable by δ) other that $< 0 >$ and R itself.

ii) R is generated as a ring by a fundamental matrix B and $det^{-1}(B)$.

Remark 3.2. *We may consider this B as a solution in $GL_m(R)$, so in fact one could restate de problem in the following way: given $A \in LGL_m(C_k) = M_m(C_k)$, we want to find $Y \in GL_m(k^{diff})$ such that $\delta Y Y^{-1} = A$. Now, as we will see in a while, this is the classical logarithmic derivative $\delta ln_G : GL_m \to LGL_m$*

Theorem 3.3.

1. *Any differential field k admits a Picard-Vessiot extension R for any linear equation, which is unique up to isomorphism if $C_k^{alg} = C_k$.*
2. *The group $G = Aut_\delta(R/k)$ of automorphisms commuting with δ is the set of C_k points of an algebraic group defined over C_k $H \leq GL_m$, whose dimension is exactly $trdeg(R/k)$.*
3. *There is a Galois correspondence between Zariski closed subgroups $H \leq G$ and differential subfields of $frac(R)$ containing k.*

We may now want to consider solutions of equations that live in an arbitrary group rather than just GL_n, a group that may not be defined over $\mathbb{C}$. To do: give a sense to δy and $exp(x)$, and find the relation between both ingredients.

Let $(k,\ \delta)$ be a differential field with field of constants C_k.

Definition 3.4. *Given an irreducible algebraic variety X[1] with $I(X) \subset k[X_1, \dots, X_n]$, for each $x \in X$ we define:*

- *i)* $T_xX = \{(u_1, \dots, u_n) : \sum_i \frac{\partial p}{\partial x_i}(x) u_i = 0\ \forall p \in I\}$ *the tangent space of X at x.*
- *ii)* $TX = \{(x, u) : x \in X\ u \in T_xX\}$ *the tangent bundle of X.*
- *iii)* $\tau_xX = \{(u_1, \dots, u_n) : \sum_i \frac{\partial p}{\partial x_i}(x) u_i + p^\delta(x) = 0 \forall P \in I\}$ *the twisted tangent space of X at x.*
- *iv)* $\tau X = \{(x, u) : x \in X\ u \in \tau_xX\}$ *the twisted tangent bundle of X.*

Remark 3.5. *If X is defined over C_k, then both spaces are equal $TX = \tau X$.*

Remark 3.6. *If X=G is a group, we may endow both TG and τG with a group structure: for each $g \in G$ consider the following maps:*

1. $\rho_g(h) = gh$

[1] If the variety is not irreducible, the definitions still make sense, but it has to be done by pieces and then paste them together in the right way. We omit this throughout the exposition. See Marker's Manin Kernels for the details.

2. $\lambda_g(h) = hg$

Now $\rho_g,\ \lambda_g,\ : G \to G$*, so their differentials* $d\rho_g,\ d\lambda_g : TG \to TG$*. In particular, since* $\lambda_{g^{-1}}(g) = \rho_{g^{-1}} = e$*, we have* $d_g\rho_{g^{-1}},\ d_g\lambda_{g^{-1}} : T_gG \to T_eG = LG$*. Write* $m(g,h) = gh = \lambda_g(h) = \rho_h(g)$ *for the group law in* G*. Then we have that* $d_{(g,h)}m(u,v) = d_h\lambda_g(v) + d_g\rho_h(u)$ *(which follows from the isomorphism* $T(G \times G) \cong TG \oplus TG$*). We may thus define* $(g,u)(h,v) = (gh, d_h\lambda_g(v) + d_g\rho_h(u))$

In a similar way, for τX*, one sets* $(g,u)(h,v) = (gh, d_{(g,h)}m(u,v) + m^\delta(u,v)) = (gh, d_h\lambda_g(v) + d_g\rho_h(u) + m^\delta(u,v))$

The tangent bundle may actually be seen as a semidirect product of G and LG. In fact, it is the action of G on H which gives the decomposition. Lets see an example: as we said above $d_g\lambda_{g^{-1}} : T_gG \to LG$. Define

$$\begin{aligned} \eta : TG &\to LG \\ (g,u) &\mapsto d_g\lambda_{g^{-1}}u \end{aligned}$$

It is a crossed group homomorphism (i.e. $\eta((x,a)(y,b)) = \eta(x,a) + (x,a)^{-1}\eta(y)(x,a) = \eta(x,a) + \eta(y,b) \star y$, where the action of TG on LG is the one induced by G). Indeed:

$$\begin{aligned} \eta((g,u)(h,v)) &= \eta(gh, d_h\lambda_g(v) + d_g\rho_h(u)) = d_{gh}\lambda_{(gh)^{-1}}(d_h\lambda_g(v) + d_g\rho_h(u)) = \\ &= d_{gh}\lambda_{(gh)^{-1}}d_h\lambda_g(v) + d_{gh}\lambda_{(gh)^{-1}}d_g\rho_h(u)) = \\ &= d_h(\lambda_{(gh)^{-1}} \circ \lambda_g)(v) + d_g(\lambda_{h^{-1}g^{-1}})\rho_h)(u) = \\ &= d_h\lambda_{h^{-1}}(v) + d_g(\lambda_{h^{-1}} \circ \lambda_{g^{-1}} \circ \rho_h)(u) = \\ &= d_h\lambda_{h^{-1}}(v) + d_g(\lambda_{h^{-1}} \circ \rho_h \circ \lambda_{g^{-1}})(u) = \\ &= d_h\lambda_{h^{-1}}(v) + d_g(\lambda_{h^{-1}} \circ \rho_h \circ \lambda_{g^{-1}})(u) = \\ &= d_h\lambda_{h^{-1}}(v) + d_e i_h d_g\lambda_{g^{-1}}(u) = \eta(h,v) + \eta(g,u) \star h \end{aligned}$$

Then $(g,u) \mapsto (g, \eta(g,u))$ defines a group homomorpism between TG and $LG \rtimes_\eta G$.

What if we had chosen a different action $LG \curvearrowleft G$? Then we would have found a different (k-rational) splitting of TG as a semidirect product of G and LG; any such splitting is given by a crossed group homomorphism $h : TG \to LG$, or equivalently, a homomorphic section $s : G \to TG$[2]

[2]The equivalence is the following: given s, we define $h_s : (g,u) \mapsto \eta(g, u - s(g))$; if we start off from a crossed homomorphism h, take $s(g) = -v_g$ where $h(g,0) = (g, v_g)$). It is a homomorphic section, since $TG \cong LG \ltimes_h G$, so when comopsing $(g,0) \mapsto (g, h(g,0))$ with the projection $\pi : TG \twoheadrightarrow G$ we define a homomorphic section $s : G \to TG$.

To bring in the differential structure of the group, which we need to do to solve the equations, consider again a group defined over the constants and the group embedding:

$$\begin{aligned} \nabla : G &\rightarrow TG \\ g &\mapsto (g, \delta g) \end{aligned}$$

By composing with one of the previously discussed h_s we obtain the logarithmic derivative associated to s:

$$\begin{aligned} \delta ln^s_G = h_s \circ \nabla : G &\rightarrow LG \\ g &\mapsto h(g, \delta g) \end{aligned}$$

Some remarks/facts:

1. If $s = 0$, we obtain $\delta ln^s_G(g) = \delta(g)g^{-1}$.
2. δln_G is surjective if k is differetnially closed (it follows from the fact that, under the given hypothesis, $\nabla(G)$ is dense in TG)
3. $ker(\delta ln_G) = G(C_k)$

Note that if instead of being defined over C_k G was defined over k, the map ∇ would still make sense, but it taking its values in τG rather than TG. When this happens, even if TG and τG are not isomorphic as algebraic groups, there are still isomorphic in a differential way.

Lemma 3.7. $(g, u) \mapsto (g, \delta g - u)$ *defines a differential rational isomorphism between* τG *and* TG*, which is rational on the fibers* $\tau_g G$*. In particular* $\tau_e G \cong T_e G = LG$*. Therefore the exact sequence* $0 \rightarrow \tau_e G \rightarrow \tau G \rightarrow G \rightarrow 0$ *determines a splitting* $0 \rightarrow T_e G \rightarrow \tau G \rightarrow G \rightarrow 0$

So as before, we may wonder in what ways τG may be written as a semidirect product of G and LG

Definition 3.8. *Let* G *be an algebraic group defined over* k*. An algebraic* D*-group structure on* G *is given by a* k*-rational homomorphic section* $s : G \rightarrow \tau G$*. We note it* (G, s)*.*

As before, composing $h_s \circ \nabla$ we obtain

$$\begin{aligned} \delta ln_G = h_s \circ \nabla : G &\rightarrow LG \\ g &\mapsto h_s(g, \delta g) \end{aligned}$$

a logarithmic derivative map.

Under an extra assumption, we may develop a Galois theory with the same properties as in the classical case. This assumption is equivalent to the no-new-constants property of Picard-Vessiot extensions (if the constants are algebraically closed, then $C_R = C_k$ for R/k Picard Vessiot.)

Definition 3.9. *A D-group (G,s) defined over k is called K-large (for a field K) if* $G^\delta(K^{diff}) = G^\delta(K)$ *for some (any) differential closure* K^{diff} *of* K.

Theorem 3.10. *Let (G,s) be a k-large D-group defined over k, and let* $L = k(\alpha)$ *for* α *a solution of* $\delta ln_G^s = a$ $(a \in LG(k))$. *Then* $Aut_\delta(L/k) = H\delta(k)$ *for some* $(H,s) \leq (G,s)$ *defined ove k. Furthermore,* $tdeg(k(\alpha)/k) = dimH$

4 Towards an extension of Lindemann-Weierstrass

This classical theorems asserts:

Theorem 4.1. *Suppose that* $x_1, .., x_n \in \mathbb{Q}^{alg}$ *(the algebraic closure of* $\mathbb{Q}$*) are* $\mathbb{Q}$*-linearly independent; then the complex numbers* $e^{x_1}, ..., e^{x_n}$ *are algebraically independent, that is* $tr.deg(\mathbb{Q}(e^{x_1}, .., e^{x_n})/\mathbb{Q}) = n$.

Let us put it in geometric terms: we may consider $(x_1, .., x_n)$ as an element of $\mathbb{C}^n = \mathbb{G}_a(\mathbb{C}) = L\mathbb{G}_m(\mathbb{C})$, and so $(e^{x_1}, .., e^{x_n}) \in (\mathbb{C}^\times)^n = \mathbb{G}_m(\mathbb{C})$. Now, the algebraic subgroups of $(\mathbb{C}^\times)^n$ are defined by systems of equations:

$$y_1^{k_1} \cdot .... \cdot y_n^{k_n} = 1$$

Moreover if H is defined by such an equation then LH is defined by $k_1x_1 + ... + k_nx_n = 0$. Hence the $\mathbb{Q}$-linearity hypothesis in Lindemann's theorem can be restated as: $x = (x_1, .., x_n) \notin L(H)$ for every proper algebraic subgroup H of $(\mathbb{C}^\times)^n$.

By what has been said in section 1, this *exp* makes sense for any commutative algebraic group defined over $\mathbb{C}$ and its Lie algebra. This points out two possible generalizations: (a) by considering "functions" $x_1(t), .., x_n(t)$ in place of algebraic numbers, and (b) by considering other commutative algebraic groups, even defined over $\mathbb{C}(t)^{alg}$, in place of $((\mathbb{C}^\times)^n, \times)$.

4.1 Functions instead of numbers

We start off from the torus $\mathbb{G}_m$. Let $K = \mathbb{C}(t)$, then a point $x = (x_1, .., x_n) \in \mathbb{G}_a^n(K)$ is simply an n-tuple of rational functions $(x_1(t), .., x_n(t))$ and then $exp(x)$ is an n-tuple of meromorphic functions (or meromorphic functions on some open set) e^{x_i} where $e^{x_i}(t) = e^{x_i(t)}$. We are interested in the transcendence degree of $K(exp(x)/K)$.

- The natural hypothesis is that $x_1, .., x_n \in K$ are $\mathbb{Q}$-linearly independent modulo $\mathbb{C}$, which is equivalent to:

- There is no proper algebraic subgroup H of $\mathbb{G}_m^n$ such that $x \in LH(K) + \mathbb{G}_a^n(\mathbb{C})$.

- The conclusion should be that $tr.deg(K(exp(x))/K) = n$, namely that e^{x_1},..,e^{x_n} are algebraically independent over K.

- This functional L-W statement is TRUE, i.e. the conclusion does follow from the hypothesis.

Remark 4.2. *Why do we allow the indefinition by constants? Clearly if $\sum_i a_i x_i = b \in \mathbb{C}$ for some $a_i \in \mathbb{Q}$, then $\prod_i (e^{x_i})^{a_i} = e^{\sum_i a_i x_i} \in \mathbb{C}$ and so the e^{x_i} are certainly not algebraically independent over K. This becomes even clearer when we state the differential analogue of the statement.*

- Moreover the result follows from (or is equivalent to) a differential algebraic statement which we briefly explain.

- Let $K = \mathbb{C}(t)$ and $L > K$ a field of meromorphic functions containing $exp(x)$, and make it a differential field by putting $\partial = d/dt$.

- Note that $\partial(e^{x_i}) = e^{x_i}\partial(x_i)$.

- Hence e^{x_i} is a solution of $\partial(y_i)/y_i = \partial(x_i)$.

- The map taking $(y_1, .., y_n) \in \mathbb{G}_m^n(L)$ to $(\partial(y_1)/y_1, ..., \partial(y_n)/y_n) \in \mathbb{G}_a^n(L)$ is the standard "logarithmic derivative" on $\mathbb{G}_m^n(L)$, written as $\partial ln_{\mathbb{G}_m^n}$.

- So the functional L-W theorem follows from the statement (special case of Ax's theorem, but also proved by Kolchin):

- Suppose (L, ∂) is a differential field extending $(K, d/dt)$, $x \in \mathbb{G}_a^n(K)$ is such that $x \notin LH + \mathbb{G}_a(\mathbb{C})$ for any proper algebraic subgroup H of $\mathbb{G}_m^n$. Suppose that $y \in \mathbb{G}_m^n(L)$ satisfies $\partial ln_{\mathbb{G}_m^n}(y) = \partial(x)$. THEN $tr.deg(K(y)/K) = n$.

4.2 Constant and nonconstant groups

We now bring in the other aspect (b) of our desired generalization. We will find several difficulties at this stage: first of all, for G defined over an arbitrary differential field, $\delta(g)$ doesn't live in TG anymore, but rather in τG. To restate the theorem in a convenient way, we would also like to be able to link the exponential and the lie algebre of the group. A way to do it is to find the right differential equation connecting them, as shown in section 1.

- The first level of generality is to consider arbitrary complex commutative algebraic groups G together with their Lie algebra (tangent space at identity) LG, the exponential map $exp : LG(\mathbb{C}) \to G(\mathbb{C})$, as well as the appropriate logarithmic derivative ∂ln_G from $LG(F) \to G(F)$ for any differential field containing $\mathbb{C}$.

- When $G/\mathbb{C}$, exp_G is the unique analytic homomorphism from $LG(\mathbb{C}) \to G(\mathbb{C})$ whose differential at 0 is the identity $id_{LG(\mathbb{C})}$. The appropriate logarithmic derivative, defined by Kolchin, again gives the differential equations satisfied by exp.

- Among the new groups entering the picture are *abelian varieties*: commutative complex algebraic groups whose underlying variety is projective. As complex Lie groups they are *compact*.

- A general commutative algebraic group G fits into a short exact sequence:

- $0 \to L \to G \to A$ where A is an abelian variety, and L is a the direct product of a "vector group" $\mathbb{G}_a^{n_1}$ and an "algebraic torus" $\mathbb{G}_m^{n_2}$.

- The straight analogue of L-W for powers of an elliptic curve defined over $\mathbb{Q}$ with complex multiplication is known but not much more.

- The functional analogue of L-W (in the differential version) for commutative algebraic groups defined over $\mathbb{C}$ and with no vectorial quotients[3], is again true. (Ax, Bertrand, Kirby.)

- There are infinite "moduli spaces" for abelian varieties, that is there are really moving families of abelian varieties, in contrast to linear commutative algebraic groups[4] For example, an elliptic curve is determined up to isomorphism, by its j-invariant which can be any element of the underlying field.

[3] A group G has no vectorial quotients if there is no surjective map $G \twoheadrightarrow \mathbb{G}_a$

[4] Indeed, any such a group is of the form $\mathbb{G}_a^m \times \mathbb{G}_m^l$, so the numbers l and n determine it up to isomorphism; but our family would be uncountable, whereas the possible nonisomorphic groups are only countable

- So this immediately suggests trying to formulate and prove functional analogues of L-W where the relevant commutative algebraic group is defined over say $\mathbb{C}(t)$ or its algebraic closure, and is not necessarily isomorphic to a group defined over $\mathbb{C}$.

- A (commutative, connected) algebraic group G say, defined over $K = \mathbb{C}(t)$ can be viewed as the "generic fibre" of an algebraic family of complex algebraic groups G_t parametrized by the affine line over $\mathbb{C}$.

- Likewise, if it is defined over $K = \mathbb{C}(S)$ for S a curve then we are talking about a family parametrized by S or at least an open subset.

- In any case we have an algebraic family $\mathbf{G}(\mathbb{C}) \to S(\mathbb{C})$ of commutative complex algebraic groups G_s say, and the "generic fibre" is G. Can we again make sense of $exp_G(x)$ for $x \in G(K)$?

- For $s \in S$ we have $exp_{G_s} : LG_s(\mathbb{C}) \to G_s(\mathbb{C})$.

- $x \in LG(K)$ "is" or gives rise to a rational section $x : S(\mathbb{C}) \to L(\mathbf{G})(\mathbb{C})$.

- For a suitable small open $U \subseteq S$ (in Euclidean topology), x gives rise to an analytic section $x : U \to L(\mathbf{G})_U$, and by $exp_G(x)$ we mean the analytic section $U \to \mathbf{G}_U$ which takes s to $exp_{G_s}(x(s))$.

- Note that $exp_G(x)$ lies in the field L of meromorphic functions on U (which naturally contains K)

- Before stating our main theorem, we will give a corollary of it:

Theorem 4.3. *Suppose that A is an abelian variety of dimension n over $K = \mathbb{C}(t)^{alg}$. Let A_0 be the $\mathbb{C}$-trace of A*[5]*. Let $x \in LA(K)$ and suppose* $\mathbf{HX}_K$*: $x \notin L(B)(K) + L(A_0)(\mathbb{C})$ for any proper abelian subvariety B of A. Let $y = exp_A(x)$. Then $tr.deg(K(y)/K) = n$.*

Note that when the $\mathbb{C}$-trace of A is 0 then the hypothesis $\mathbf{HX}_K$ is simply that $x \notin L(B)$ for any proper algebraic subgroup B of A, and we obtain a statement very similar to the arithmetic L-W theorem.

The question of what differential equations are satisfied by these "relative" exponential maps is a delicate one, related to Picard-Fuchs equations, Gauss-Manin connection,..., some of which have been discussed in Daniel's talks.

[5] That is to say the maximal subgroup $A_0 \leq A$ such that A/A_0 does not have vectorial quotients

5 The constant case

To give a flavour of what's happening, we start by the case $G = \mathbb{G}_m^n$., which uses the Galois theory of linear differential equations as a main tool:

Theorem 5.1. *Let* $K = \mathbb{C}(t)$, $G = \mathbb{G}_m^n$ *the complex n-dimensional torus. Suppose:*

i) $x \in LG(K)$, $x \notin LH(K) + LG(\mathbb{C}) = LH(K) + \mathbb{G}_a^n(\mathbb{C})$ $\forall H \lneq G$ *proper subgroup.*

ii) $y \in G(K^{diff})$ $\partial ln_G y = \partial x = a$

Then $trdeg(K(y)/K) = n$

Proof. The proof is done by induction on $n = dim(G)$. So let $n = 0$, that is, $G = \{1\}$, and the statement is then trivial.

Let thus $dim(G) > 0$, and suppose $trdeg(K(y)/K) < n$; by Galois theory $trdeg(K(y)/K) = dim_{\mathbb{C}} H$ where $H(\mathbb{C}) = Gal(K(y)/K) \lneq G(\mathbb{C})$. Quotienting by H we get $G_1 = G/H \cong \mathbb{G}_m^l$ [6] and $x_1 = x/LH \in LG_1 = LG/LH$, $y_1 = y/H \in G_1$, which satisfy

$$\frac{\delta y_1}{y_1} = \delta x_1$$

$$x_1 \notin LB(K) + \mathbb{G}_a^l(\mathbb{C}) \ \forall B \lneq G_1$$

Now, since $l < n$, by induction hypothesis, we have $l = trdeg(K(y_1)/K)$; but also $dim(Gal(K(y_1))/K) = l$. Now, any element in $Gal(K(y_1)/K)$ extends to an element in $Gal(K(y)/K)$ (indeed define $\tilde{\sigma}(y_1, \ldots, y_l, y_{l+1}, \ldots, y_n) = (\sigma(y_1, \ldots, y_l), y_{l+1}, \ldots, y_n))$; so since y_1 is fixed by $Gal(K(y)/K)$, hence a fortiori by $Gal(K(y_1)/K)$, by the Galois correspondence we must have $y_1 \in G(K)$. But as a simple analysis of poles shows, this is not possible, since if $y_1 = (g_1(t)), \ldots, g_l(t))$, $x_1 = (f_1(t), \ldots, f_l(t))$, the equations become

$$\frac{g'(t)}{g(t)} = f'(t)$$

which can't happen in any case unless $g' = f' = 0$ (which isn't true, for $x \notin LB(K) + \mathbb{G}_a(\mathbb{C})^n \ \forall \ B \lneq G$, so in particular x is not constant.) So it must be $l = n$. □

[6] This follows from the connectedness of G/H-being a quotinent of an absolutely irreducible group by an automorphism group- and the classification of connected commutative algebraic groups:any connected commutative algebraic group fits into an exact sequence $0 \to \mathbb{G}_a^l \times \mathbb{G}_m^k \to G \to A \to 0$ for some abelian variety A. For those familiar with the theory of linear algebraic groups, it follows from he fact that G/H is a connected d-group, hence a torus.

6 Outline of the proof

Statement of Theorem

Theorem 6.1. *Let $K = \mathbb{C}(t)^{alg}$ with $\partial = d/dt$. Let G be a commutative connected algebraic D-group defined over K, with no vectorial quotients (as an algebraic group). Let $x \in LG(K)$, and $y \in G(\mathcal{U})$ with $\partial ln_G(y) = \partial_{LG}(x)$. Assume*
HX: *$x \notin LH + (LG)^\partial$ for any proper algebraic subgroup H of G defined over K.*
THEN $tr.deg(K(y)/K) = n = dim(G)$.

Theorem 6.2. *Let again G be a commutative algebraic D-group defined over $K = \mathbb{C}(t)^{alg}$ with no vectorial quotients, and $\partial = d/dt$ on K. Let $x \in LG(K)$ and $y \in G(\mathcal{U})$ be such that $\partial ln_G(y) = \partial_{LG}(x)$. Assume $(\mathbf{HG})_0$[7], and*
$\mathbf{HX}_K$: *$x \notin LH(K) + (LG)^\partial(K)$ for any proper algebraic subgroup H of G defined over K.*
THEN $tr.deg(K(y)/K) = n = dim(G)$.

Descent of $(HX)_K$

G is a commutative algebraic D-group over $K = \mathbb{C}(t)^{alg}$ with no vectorial quotients, and satisfying $(\mathbf{HG})_0$: the semiconstant part of G is constant.

Lemma 6.3. *Suppose $x \in LG(K)$, and $x \notin LH(K) + (LG)^\partial(K)$ for any proper algebraic subgroup H of G defined over K. Let G_1 be a D-group quotient of G defined over K and x_1 the image of x in LG_1. Then $x_1 \notin LH_1 + (LG_1)^\partial$ for any proper algebraic subgroup H_1 of G_1 defined over K. Moreover G_1 also satisfies $(\mathbf{HG})_0$.*

Proof. Uses results of Deligne that if A is an abelian variety over K and and $A = A_0 \cdot A_1$ is its decomposition into a group A_0 over $\mathbb{C}$ and a group A_1 with $\mathbb{C}$-trace 0, and $\tilde{A}$ is the universal extension of A by a vector group, then (a) $L\tilde{A}$ (with its canonical connection) is a semisimple ∂-module, and (b) $(L\tilde{A})^\partial(K) = L\tilde{A}_0(\mathbb{C})$.

Theorem of the kernel

As usual $K = \mathbb{C}(t)^{alg}$.

Let A be an abelian variety over K with $\mathbb{C}$-trace 0. Let $\tilde{A}$ be its universal extension of A by a vector group:
$0 \to W_A \to \tilde{A} \to A \to 0$.
So $\tilde{A}$ has a unique D-group structure. W_A, although not a D-subgroup of $\tilde{A}$

[7] This is just a tachnical hypothesis about semiconstancy: it says that the semiconstant part of G is constant; for instance, it holds for any abelian variety A whose $\mathbb{C}$-trace is 0.

can be identified with a subspace of $L\tilde{A}$, and as such is its own Lie algebra. Let $V \subseteq W_A$ a D-subgroup of $\tilde{A}$ defined over K (necessarily different from W_A) and $G = \tilde{A}/V$ (also a D-group defined over K). The following is a version of Manin's theorem of the kernel, based on subsequent work of Coleman and Chai.

Lemma 6.4. *Suppose $x \in LG(K)$, $y \in G(K)$, and $\partial ln_G(y) = \partial_{LG}(x)$. Then $x \in W_A/V$.*

Proof. Daniel discussed it.

The socle theorem for DCF_0

Theorem 6.5. *Suppose G is a connected commutative group of finite Morley rank definable in DCF_0, that is, definable in our saturated model $\mathcal{U}$ of DCF_0. Let Y be a definable subset of G of Morley degree 1, or better, an irreducible Kolchin closed subset of G. Let $S = Stab_G(Z)$. Suppose that S is finite. THEN Z is contained in a coset (translate) of H where H is the maximal connected definable subgroup of G which is "internal to the constants" (i.e. definably isomorphic to a group definable in $(\mathcal{C}, +, \cdot)$).*

Note: there is a conjectural version for arbitrary groups of finite Morley rank, where "internal to the constants" is replaced by "internal to the family of nonmodular strongly minimal sets".

Restatement (or consequence) for algebraic D-groups, with PHS's in place of groups:

Theorem 6.6. *Let G be a commutative connected algebraic D-group, defined over an algebraically closed differential subfield K of $\mathcal{U}$. Let $Y \subseteq G$ be a coset (translate) of G^∂, and Z an irreducible differential algebraic subset of Y. Let $S < G^\partial$ be the stabilizer of Y (with respect to the regular action of G^∂ on Y). Suppose that S is finite. Then Z is contained in a coset (or translate or orbit) of H^∂ where H is the maximal connected isoconstant D-subgroup of G.*

Proof of theorems

- Let $dim(G) = n$.
- We fix x, and let $a = \partial_{LG}(x) \in LG(K)$.
- Let Y be the solution set of $\partial ln_G(-) = a$ (in $G = G(\mathcal{U})$ say).
- Y is coset (translate) of G^∂ in G.

- As a "generic" point of Y over K has $tr.deg\, n$ over K (why ?), it suffices to prove that
(*) Y has no proper differential algebraic subvariety, defined over K, i.e. that Y isolates a complete type over K,

- We prove this by induction on n.

- If $dim(G) = 1$ then $G = \mathbb{G}_m$ or an elliptic curve over $\mathbb{C}$, so Ax-Kirby does the job.

- Now suppose $dim(G) = n > 1$. Suppose for a contradiction that (*) fails, witnessed by a proper irreducible differential algebraic subvariety Z of Y, defined over K.

- We are in the situation of Theorem 1.6. Let $S < G^{\partial}$ be the stabilizer of Z. S is a differential algebraic subgroup of G^{∂} so of the form $S' \cap G^{\partial}$ for some algebraic D-subgroup S' of G. Note that S, S' are defined over K, and are proper subgroups of G^{∂}, G, respectively.

- **Case I.** S, and so S', is infinite.

- Let $H = G/S'$, another connected algebraic D-group over K, of dimension positive but $< n$, and let $\pi : G \to H$ be the quotient homomorphism, which is also a homomorphism of algebraic D-groups.

- π induces $L\pi : LG \to LH$. Let $x' = (L\pi)(x) \in LH(K)$ and $a' = (L\pi)(a) \in LH(K)$.

- Then $\partial_{LH}(x') = a'$ and $\pi(Y)$ is the solution space of $\partial ln_H(-) = a'$.

- The hypothesis **HX** clearly descends to the new data. Moreover assuming $(\mathbf{HG})_0$ as in Theorem 1.2, $(\mathbf{HX})_K$ is true of the new data by Lemma 1.3, and moreover the D-group H also satisfies $(\mathbf{HG})_0$.

- Hence in either the context of Theorem 1.1 or 1.2, we see, by induction hypothesis, that $\pi(Y)$ "isolates a complete type over K". In particular $\pi(Z) = \pi(Y)$, whereby $Y \subseteq Z + S'$, so $Y = Z + S$.

- But S stabilizes Z, so $Y = Z$, contradiction.

- **Case II.** S is finite.
- By Theorem 1.6, Z is contained in a translate, necessarily defined over K, of H^∂, where H is the maximal connected isoconstant D-subgroup of G.
- Let now $\pi : G \to G/H$ (a D-group) be the quotient map, and as in Case I, we obtain $\pi(x) = x' \in L(G/H)(K)$, $\pi(a) = a' \in L(G/H)(K)$, where $\partial_{L(G/H)}(x') = a'$, and $\pi(Y)$ is the solution set of $\partial ln_{G/H}(-) = a'$.
- But now $\pi(Z)$ is a K-rational point, say y' of $\pi(Y)$. Moreover the hypotheses **HX** and, where appropriate $(\mathbf{HX})_K$ and $(\mathbf{HG})_0$, are preserved.
- We now have three subcases:
- **Subcase II(a)**. $H = G$. This says that G is isocontant, and so isomorphic over K to a group defined over $\mathbb{C}$ with trivial D-structure ($s = 0$). So we are in the constant case, where the conclusion of the Theorem is known (Ax-Kirby-Bertrand), giving us either the desired conclusion or a contradiction.
- **Subcase II(b)**. H is a nontrivial proper subgroup of G. But then $dim(G/H) < n$ is positive, and our K-rational point y' with $\partial ln_{G/H}(y') = \partial_{L(G/H)}(x')$ contradicts the induction hypothesis.
- **Subcase II(c)** $H = \{0\}$. So G has NO nontrivial isoconstant D-subgroups. One can conclude with a bit of work that G is of a special form: it's semiabelian part A is abelian and traceless.
- Namely, with notation as just before Lemma 1.4, there is an abelian variety A over K with $\mathbb{C}$-trace 0 such that G is a quotient of $\tilde{A}$ by a D-subgroup V of W_A.
- Note that Z is now a singleton $\{y\}$, with $y \in G(K)$, and $\partial ln_G(y) = \partial_{LG}(x)$.
- By Lemma 1.4, $x \in W_A/V$. But W_A/V is the Lie algebra of the algebraic subgroup W_A/V of G. This contradicts the hypothesis $(\mathbf{HX})_K$.
- The proof is complete.