Lindemann-Weierstrass for semi-abelian varieties over function fields

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Lindemann-Weierstrass

• $K = \mathbb{Q}^{alg}, G = \mathbb{G}_m^n, LG := Lie(G) \simeq (\mathbb{G}_a)^n$
• $x = (x_1, ..., x_n) \in LG(K) = (\mathbb{Q}^{alg})^n$, such that
(HX) : $x \notin LH$ for any proper algebraic subgroup H of G;
• $y = exp_G(x) = (e^{x_1}, ..., e^{x_n}) \in G(\mathbb{C}) \simeq (\mathbb{C}^*)^n$
Then, $tr.deg.(K(y)/K) = dimG$.
NB : condition on x, not on y ($\Rightarrow \pi \notin \mathbb{Q}^{alg}$).

(A corollary of) Ax

• $K = \mathbb{C}(z)^{alg}, \partial = \frac{d}{dz}$;
• G: commutative algebraic group *defined over* $\mathbb{C}$, with no additive quotient;
• so, $LG/\mathbb{C}$. Let $x \in LG(K)$, with
$(HX)_K$: $x \notin LH(K) + LG(\mathbb{C})$, for any proper algebraic subgroup H/K of G.
• $y =$ "exp_G" $(x) \in G(K^{dif})$, meaning

$$\partial \ell n_G(y) = \partial_{LG}(x);$$

Then, $tr.deg.(K(y)/K) = dimG$.

Here

We allow G **non-constant**, i.e. G/K.

For instance,
• $G = A/K$, an abelian variety; we extend it to an abelian scheme $\pi : \mathbf{A} \to S$ over some smooth algebraic curve $S/\mathbb{C}$.
• $(A_0, \tau) = K/\mathbb{C}$-trace of A; we'll drop τ and view A_0 as the "constant part" of A.
• $x \in LA(K)$, extended to $\mathbf{x} \in L\mathbf{A}(S)$.
• Consider the exact exponential sequence of analytic sheaves over S :

$$0 \to \mathcal{P} \to L\mathbf{A}^{an} \longrightarrow \mathbf{A}^{an} \to 0,$$

where $\mathcal{P}$ is the local system of periods (dual of $R^1\pi_*(\mathbb{Z})$) on S;
• *Define* $\mathbf{y} = exp_{\mathbf{A}}(\mathbf{x}) \in \mathbf{A}^{an}(S)$.

Theorem 1. (B-P) *Let* $x \in LA(K)$, *with* $(HX)_K$: $x \notin LH(K) + LA_0(\mathbb{C})$ *for any proper abelian subvariety* H *of* A.
Then, $tr.deg.(K(\mathbf{y})/K) = dim(\mathbf{A}/S)$.

NB 1 : again, condition on x, not y (but here, the proof will appeal to the known fact that the non-constant periods are transcendental).
NB 2 : The arithmetic analogue is unknown.

DAG, ADG, DM

K : a function field in one variable over $\mathbb{C}$ (or its algebraic closure), ∂ : a derivation on K with constants $K^\partial = \mathbb{C}$.

DM : D-**modules** ($= K[\partial]$-modules of finite length); equivalently, vectorial D-groups.

DAG : (commutative) **differential algebraic groups** over K (see C. Wood, LNM 1696). For instance, $A^\sharp =$ kernel of Buium's map $\mu : A \to (\mathbb{G}_a)^N$. By Pillay, $A^\sharp(K^{dif}) = A^\sharp(K)$.

ADG : (commutative) **algebraic D-groups** over K: an algebraic group G/K, equipped with an extension of ∂ to a derivation on $K(G)$ respecting the group structure. Equivalently, a K-rational splitting of the *twisted tangent bundle* $T_\partial G \in Ext(G, LG)$:

$$s : G \to T_\partial G.$$

If $G := (G, s)$ is an ADG over K, its **logarithmic derivative**:

$$\partial\ell n_G : G \to LG : y \mapsto \partial y - s(y)$$

is a first order differential homomorphism. Its kernel $G^\partial = \{y \in G, \partial y = s(y)\}$ is a DAG over K of "finite dimension", Zariski dense in G.

• At the Lie algebra level, the ADG structure on G induces on the vectorial group LG a canonical ADG, i.e. DM, structure, whose logarithmic derivative is a connexion

$$\partial\ell n_{LG} = d_0 \partial\ell n_G := \partial_{LG} : LG \to LG.$$

We write $(LG)^\partial$ for the DAG group of its horizontal sections.

Let K^{dif} be a differential closure of K, and let $x \in LG(K^{dif})$, $y \in G(K^{dif})$. We will check that such analytic relations as

$$y = \text{``}exp_G\text{''}(x) \text{ ,or } x = \text{``}\ell n_G\text{''}(y)$$

can be meaningfully replaced by the differential relation

$$\partial\ell n_G(y) = \partial_{LG}(x) \quad (\dagger).$$

Transcendence ... and Galois groups

Given $x \in LG(K)$, set $a = \partial_{LG}(x) \in LG(K)$. The differential relation (†) becomes a differential equation

$$\partial \ell n_G(y) = a \quad (*),$$

in the unknown $y \in G(K^{dif})$, and a solution y provides a differential extension $K(y)$ of K. If G were the (constant, non commutative) group GL_n, with $\partial \ell n_G(y) = (\partial y)y^{-1}$, this would just be a Picard-Vessiot extension, with transcendence degree given by the dimension of its Galois group $H \subset GL_n$ over $\mathbb{C}$.
Now, Pillay (Pacific J. Maths, 216, 2004, 343-360 = [Pi]) has shown that under the "no new constants" assumption that

- the ADG G is K-*large*: $G^{\partial}(K^{dif}) = G^{\partial}(K)$,

one can attach to (*) an ADG subgroup H/K of G sharing exactly the same properties, viz.:

$$Aut_{\partial}(K(y)/K) \simeq H^{\partial}(K);$$

$$tr.deg(K(y)/K) = dim H.$$

To prove Theorem 1, we can then follow the proof of Kolchin's classical theorem on the toric case $G = (\mathbb{G}_m)^n$: if $H \neq G$, we go to the quotient $G_1 = G/H$ and find

$$x_1 \in LG_1(K), y_1 \in G_1(K)$$

such that $\partial \ell n_{G_1}(y_1) = \partial_{LG_1}(x_1)$.

It remains to show that an exact differential form can equal an exact logarithmic differential form only if it vanishes.
The answer (given, say, by residues in the toric case, or more generally, by Ax's arguments on differentials when G is constant - see also Kirby's and Kowalski's lectures) will follow from *Manin's kernel theorem* in the case of a general ADG over K.

The twisted tangent bundle

- $K = \mathbb{C}(S)$; $\partial \in H^0(S, TS)$: a vector field;
- $\pi : \mathbf{G} \to S$: a group scheme; $\mathbf{e} =$ 0-section,
- $L\mathbf{G}$: the pull-back $\mathbf{e}^*(T_{\mathbf{G}|S})$ of the relative tangent bundle of $\mathbf{G}$ over S.
- At the generic point of S, we get G/K, with (relative) tangent bundle $TG \simeq G \times LG$
- The (full) tangent bundle $T\mathbf{G}$ of $\mathbf{G}$ sits in an exact sequence

$$0 \to T_{\mathbf{G}/S} \to T\mathbf{G} \to \pi^*(TS) \to 0$$

of vector bundles over $\mathbf{G}$, and is also a group scheme over TS. When t runs through S, its fibers $(T\mathbf{G})_{(t,\partial_t)}$ yield a group scheme $T_\partial \mathbf{G}$ over S, whose generic fiber is the **twisted tangent bundle** $T_\partial G/K$.
- A section $\mathbf{x}$ of $\mathbf{G}/S$ provides a section $\mathbf{x}_*(\partial)$ of $T_\partial \mathbf{G}/S$, written ∂x at the generic point.
- Viewed over K, $T_\partial G$ is a *group extension of G by LG*;
- Viewed over G, $T_\partial G$ is a *torsor under LG*, whose class in $H^1(G, TG)$ is given, in the proper case, by the **Kodaira-Spencer** map.

Almost semi-abelian D-groups

Suppose $G = A$, an abelian variety of dimension g, and consider the g-dim'l vector group

$$W_A = K - \text{dual of } H^1(A, \mathcal{O}_A).$$

Then the torsor $T_\partial A$ represents the (image of ∂ under the) Kodaira-Spencer map

$$\kappa(\partial) \in H^1(A, \underline{TA}) = H^1(A, \mathcal{O}_A) \otimes L_A = Hom(W_A, L_A)$$

A section s exists iff $\kappa(\partial) = 0$, equivalently: iff $A = A_0$ is (iso-)constant, as in Ax's case. So, in general, we are **forced** to introduce the *universal vectorial extension*

$$\tilde{A} \in Ext(A, W_A) :$$

by its very definition, there is a section s of $T_\partial \tilde{A} \to \tilde{A}$, and indeed a unique one, so that $\tilde{A}$ has a canonical ADG structure.

As soon as $\kappa(\partial) \neq 0$, W_A is *not* an algebraic D-subgroup of $\tilde{A}$. Denote by U_A the maximal ADG subgroup of $\tilde{A}$. Then, $\tilde{A}/U_A$ again has a canonical ADG structure, and the projection to A induces a DAG isomorphism

$$(\tilde{A}/U_A)^{\partial} \simeq A^{\sharp}.$$

More generally :

• $G \in Ext(A,T)$: a semi-abelian variety defined over K; $W_A =$ dual of $H^1(A, \mathcal{O}_A)$;

• $\tilde{G} \in Ext(G, W_A)$: the universal vectorial extension of G, with canonical ADG structure.

• Choose $U \subset U_A \subset W_A$: a vectorial D-subgroup of $\tilde{G}$.

Finally, define an **almost semi-abelian** D-**group** as the quotient $\mathcal{G} = \tilde{G}/U$ of any such $\tilde{G}$ by any such U. It has a canonical ADG structure, a maximal vector subgroup $\mathcal{W}$, etc.

NB : the class of almost semi-abelian D-groups over K is closed under D-quotients, but *not* under D-subgroups.

The theorems

Let $\mathcal{G}$ be an almost semi-abelian D-group defined over K. Let $x \in L\mathcal{G}(K)$, and let $y \in \mathcal{G}(K^{dif})$ be a solution of

$$\partial \ell n_{\mathcal{G}}(y) = \partial_{L\mathcal{G}}(x). \qquad (*)$$

Theorem 2. *: assume that*
(HX) *:* $x \notin LH + (L\mathcal{G})^{\partial}$ *for any proper algebraic subgroup* H *of* $\mathcal{G}$ *defined over* K*.*
Then, $tr.deg.(K(y)/K) = dim\mathcal{G}$*.*

Theorem 3. *: assume that*
$(HX)_K$*:* $x \notin LH(K) + (L\mathcal{G})^{\partial}(K)$ *for any proper algebraic subgroup* H *of* $\mathcal{G}$ *defined over* K*, and that*
(HG_0) *: the semi-constant part (see below) of* $\mathcal{G}$ *is actually constant.*
Then, $tr.deg.(K(y)/K) = dim\mathcal{G}$*.*

NB : Theorem 3 $\Rightarrow$ Theorem 1.

Constant parts

• Let (A_0, τ) = the $K/\mathbb{C}$-trace of A. The *semi-constant part* of G (or: of $\mathcal{G}$) is the pull-back $G_0 \in Ext(A_0, T)$ of G to A_0.
• Let G_0^0 be the maximal constant quotient of G_0. So, G^0 is constant iff $G_0 = G_0^0$.
• As already mentioned, the following "no new constants" notion is basic to Pillay's differential Galois theory: the ADG group $\mathcal{G}$ is K-**large** if

$$\mathcal{G}^\partial(K^{dif}) = \mathcal{G}^\partial(K).$$

Let $A/\tau A_0 = A_1$ be the maximal traceless quotient of A. Then,

Lemma 4. *The almost semi-abelian D-group $\mathcal{G} = \tilde{G}/U$ is K-large if and only if*
(i)= $(HG_0) : G_0 = G_0^0$, and
(ii) U projects onto the maximal vectorial D-subgroup U_{A_1} of W_{A_1}.

NB : the semi-abelian varieties G not satisfying (HG_0) lead, in more general terms, to the study of an interesting new map

$$\xi : Ext(A, T) \to Ext_{ADG}(U_A, T).$$

Gauss-Manin

$$\partial \ell n_{\mathcal{G}} = \partial_{L\mathcal{G}} \circ \ell n_{\mathcal{G}}$$

Following Manin, we now give an analytic description of $\partial \ell n_{\mathcal{G}}$ in terms of the group scheme $\mathbf{G}$ extending $\mathcal{G}$ over the Riemann surface S. Again, we have the exponential exact sequence

$$0 \to \mathcal{P} \to L\mathbf{G}^{an} \to \mathbf{G}^{an} \to 0 \ .$$

Over sufficiently small balls $\mathcal{U}$ of S, $exp_{\mathbf{G}}$ can be inverted into a "multivalued logarithm"

$$\ell n_{\mathbf{G}^{an}_{|\mathcal{U}}} : \mathbf{G}^{an}(\mathcal{U}) \to L\mathbf{G}^{an}(\mathcal{U})/\mathcal{P}(\mathcal{U}).$$

The *Gauss-Manin connection* $\nabla_{L\mathbf{G}}$ on the $\mathcal{O}_S$-module $L\mathbf{G}$ is the unique connection admitting the image of the local system $\mathcal{P} \otimes \mathbb{C} = (R^1\pi_*\mathbb{C})^*$ as space of horizontal sections. It is actually *algebraic*, and it can be shown that the connection $\nabla_{L\mathcal{G}}$ it induces at the generic point of S satisfies

$$\nabla_{L\mathcal{G}}(\partial) = \partial_{L\mathcal{G}}.$$

Since the section of $\mathcal{P}$ are annihilated by $\partial_{L\mathbf{G}}$, the morphism of analytic sheaves of abelian groups

$$\lambda := \partial_{L\mathbf{G}} \circ \ell n_{\mathbf{G}} : \mathbf{G}^{an} \to L\mathbf{G}^{an}$$

is well-defined. Moreover, λ preserves moderate growth, and therefore induces a homomorphism $\lambda_K : \mathcal{G}(K) \to L\mathcal{G}(K)$, which actually shares all the properties characterizing $\partial \ell n_{\mathcal{G}}$. From this (or more directly, from the study of $exp_{\mathbf{G}}$), one finally deduces that for a section $\mathbf{x} \in L\mathbf{G}(S)$ extending $x \in L\mathcal{G}(K)$,

$$\mathbf{y} = exp_{\mathbf{G}}(\mathbf{x}) \Rightarrow \partial \ell n_{\mathcal{G}}(y) = \partial_{L\mathcal{G}}(x),$$

where $\mathbf{y} \in \mathbf{G}^{an}(S)$ and $y \in \mathcal{G}(K^{dif})$ generate over K isomorphic differential extensions.

• This shows that Theorem 3 does imply Theorem 1 (use $(L\tilde{A})^{\partial}(K) = L\tilde{A}_0(\mathbb{C})$, and the fact that $\tilde{A}$ is an essential extension).

• It further shows that $\partial_{L\mathcal{G}}$ and $\partial \ell n_{\mathcal{G}}$ coincide on the maximal vectorial subgroup $\mathcal{W}$ of $\mathcal{G}$, since the exponential map of such groups is the identity:

$$\partial \ell n_{\mathcal{G}}|_{\mathcal{W}} = \partial_{L\mathcal{G}}|\mathcal{W}.$$

Manin maps and Manin kernel

• $\mathcal{G} \in Ext(G, \mathcal{W})$: an almost semi-abelian D-group over K.
• $Ext_{DM/K}(\mathbf{1}, L\mathcal{G}) \simeq L\mathcal{G}(K)/\partial_{L\mathcal{G}}(L\mathcal{G}(K))$: the $\mathbb{C}$-vector space of D-module (DM) extensions of $\mathbf{1} = (K, \partial)$ by $(L\mathcal{G}, \partial_{L\mathcal{G}})$.
Following Manin, we define a map

$$M_K : G(K) \to Ext_{DM/K}(\mathbf{1}, L\mathcal{G})$$

by attaching to a point $\overline{y} \in G(K)$ the projection $M_K(\overline{y})$ of $\partial\ell n_{\mathcal{G}}(y)$ in $L\mathcal{G}(K)/\partial_{L\mathcal{G}}(L\mathcal{G}(K))$, where y is any lift of $\overline{y}$ to $\mathcal{G}(K)$ - this is well defined, since $\partial\ell n_{\mathcal{G}}|_{\mathcal{W}} = \partial_{L\mathcal{G}}|_{\mathcal{W}}$. So, $M_K(\overline{y})$ is the class of the extension defined by the inhomogeneous linear differential equation

$$\partial_{L\mathcal{G}}(x) = \partial\ell n_{\mathcal{G}}(y) \quad (**)$$

(with x as unknown !) which occurs in the study of the "logarithmic (or Grothendieck) case" of the Schanuel conjecture.

Proposition 5. (Manin-Coleman-Chai)*: for $G = T \times A$, the kernel of M_K coincides with the divisible hull of $G_0(\mathbb{C})$ in $G(K)$.*

In other words, for $G = T \times A$, the differential equation $(**)$ admits a K-rational solution $x \in L\mathcal{G}(K)$ if and only if $\overline{y} \in G(K)$ lifts in $\mathcal{G}(K)$ to a point $y \in \mathcal{W}(K) + \mathcal{G}^{\partial}(K)$.

The above map M_K should not be confused with the DAG map (of high order)

$$\mu : G \to L\mathcal{G}/\partial_{L\mathcal{G}}(\mathcal{W}) \hookrightarrow (\mathbb{G}_a)^N,$$

which attaches to a point $\overline{y} \in G$ the projection $\mu(\overline{y})$ of $\partial\ell n_{\mathcal{G}}(y)$ in $L\mathcal{G}/\partial_{L\mathcal{G}}(\mathcal{W})$, for any lift $y \in \mathcal{G}$ of $\overline{y}$ (well-defined for the same reason). Contrary to M_K, μ is of a "geometric" nature (i.e. insensitive to base extension).

Proposition 6. (Buium)*: the kernel $G^\sharp$ of μ coincides with the Kolchin closure of the torsion subgroup of G, and is isomorphic as DAG to $(\tilde{G}/U_A)^{\partial}$.*

NB : for $G = T \times A$, $\tilde{G}/U_A$ *is always a K-large ADG* (cf. Lemma 4). Combined with Chai's sharpening of Proposition 5, this says that $G^\sharp(K^{dif})$ also coincides with the divisible hull of $G_0(\mathbb{C})$ in $G(K)$, i.e.

$$(KerM_K)\text{“}(K)\text{”} = (Ker\mu)(K^{dif}) \; !$$

.

Descending $(HX)_{[K]}$ to quotients

Let $\mathcal{G}$ be an almost semi-abelian D-group over K, and let H/K be an algebraic D-subgroup of $\mathcal{G}$. Then, $\mathcal{G}_1 = \mathcal{G}/H$ has a canonical structure of almost semi-abelian D-group, and both its logarithmic derivative $\partial\ell n_{\mathcal{G}_1}$ and the Gauss-Manin connection $\partial_{L\mathcal{G}_1}$ are compatible with the quotients. In particular, if $x \in L\mathcal{G}$ satisfies Hypothesis (HX), so does its projection $x_1 \in L\mathcal{G}_1$.

As for $(HX)_K$:

Lemma 7. *Assume that the semi-constant part G_0 of $\mathcal{G}$ is actually constant, i.e. (HG_0). Let $x \in L\mathcal{G}(K)$ satisfy $(HX)_K$, and let $\mathcal{G}_1$ be a D-group quotient of $\mathcal{G}$. Then, the image x_1 of x in $L\mathcal{G}_1$ satisfies $(HX)_K$ (relatively to $\mathcal{G}_1$).*

This follows from Part (i) of

Proposition 8. (Deligne)*: let $G = T \times A$. Then,*

i) the D-module $L\tilde{G}$ is semisimple;
ii) $(L\tilde{G})^{\partial}(K)$ reduces to $L\tilde{G}_0(\mathbb{C})$.

Galois theoretic proof of Theorem 3.

• *The K-large case.*
We first treat the case where $\mathcal{G}$ is K-large, i.e. $\mathcal{G}^{\partial}(K^{dif}) = \mathcal{G}^{\partial}(K)$ (see also Lemma 4). Look at the equation $(*)$ in the shape

$$\partial \ell n_{\mathcal{G}}(y) = a, \text{ with } a = \partial_{L\mathcal{G}}(x) \in L\mathcal{G}(K).$$

Then, the field $K(y)$ generated by any of its solutions $y \in \mathcal{G}(K^{dif})$ depends only on a, and we can consider the *Galois group* $\mathcal{H}$ of the equation, in the sense of Pillay ([Pi], cf. slide # 5): recall that this is an algebraic D-subgroup of $\mathcal{G}$, defined over K, with the property that $Aut_{\partial}(K(y)/K) \simeq \mathcal{H}^{\partial}(K)$, and with a Galois correspondence. Under Hypothesis $(HX)_K$, we claim that

$$Aut_{\partial}(K(y)/K) \simeq \mathcal{G}^{\partial}(K)$$

(from which $tr.deg.(K(y)/K) = dim\mathcal{G}$ immediately follows).

Indeed, suppose the Galois group $\mathcal{H}$ of the equation does not fill up $\mathcal{G}$. We then get a similar equation $(*)_1$ on the quotient $\mathcal{G}_1 = \mathcal{G}/\mathcal{H}$, with trivial Galois group, i.e. with a K-rational solution y_1, and where x_1 again satisfies $(HX)_K$, by Lemma 7. If the abelian part of $\mathcal{G}_1$ is trivial, we conclude by Ax's theorem. Otherwise, we mod out by the toric part of $\mathcal{G}_1$, and apply Proposition 5 to the quotient $\mathcal{A}'$: since both y' and x' are K-rational, we have $y' = w' + z' \in \mathcal{W}'(K) + \mathcal{A}'^{\partial}(K)$, hence $x' - w' \in (L\mathcal{A}')^{\partial}(K)$, contradicting $(HX)_K$.

• *The general case*

For the general case, denote by $K_{\mathcal{G}}$ the subfield of K^{dif} generated by all the points of $\mathcal{G}^{\partial}(K^{dif})$. Then $\mathcal{G}$ is $K_{\mathcal{G}}$-large, and the equation admits a Galois group $\mathcal{H} < \mathcal{G}$, a priori only defined over $K_{\mathcal{G}}$, with $\mathcal{H}^{\partial}(K_{\mathcal{G}}) \simeq Aut_{\partial}(K_{\mathcal{G}}(y)/K_{\mathcal{G}})$. We proceed to prove that again, $\mathcal{H}$ fills up $\mathcal{G}$.

Since $\mathcal{G}$ satisfies (HG_0), $K_{\mathcal{G}}$ is the Picard-Vessiot extension of K generated by the horizontal vectors of $\partial_{L\mathcal{G}}$ lying in the (Lie algebra of the) maximal D-vectorial subgroup

$\mathcal{U} = U_A/U$ of $\mathcal{G}$. By Proposition 8, $J := Aut_\partial(K_\mathcal{G}/K)$ is therefore a *reductive* linear algebraic group over $\mathbb{C}$. In fact, consideration of the Betti lattice of horizontal vectors of $\partial_{L\mathcal{G}}$ shows that $J = [J, J]$ is even *semi-simple*.

Since $\mathcal{H}^\partial$ embeds in $\mathcal{G}^\partial$ via the J-equivariant map $\xi : \sigma \mapsto \sigma y - y$, $\mathcal{H}$ is actually defined over K, and we can consider the almost semi-abelian D-group $\mathcal{G}_1 = \mathcal{G}/\mathcal{H}$ over K. The corresponding equation $(*)_1$ now has a solution y_1 defined over $K_\mathcal{G}$, i.e. its $\mathcal{G}_1^\partial$-torsor of solutions $\mathcal{Y}_1$ is trivial over $K_\mathcal{G}$. Denoting by $\mathcal{U}_1$ the maximal D-vectorial-subgroup of $\mathcal{G}_1$, so that $(\mathcal{G}_1/\mathcal{U}_1)^\partial \simeq G_1^\sharp$ (cf. Proposition 6), we have an exact sequence

$$H^1(J, \mathcal{U}_1^\partial) \to H^1(J, \mathcal{G}_1^\partial) \to H^1(J, G_1^\sharp),$$

whose first term vanishes by reductivity; its last term also vanishes, since it identifies with $Hom(J, G_1^\sharp)$, while J has no commutative quotient. Therefore, the torsor $\mathcal{Y}_1$ is trivial over K, i.e. the equation $(*)_1$ has a K-rational solution. As in the K-large case, we conclude that $\mathcal{G}/\mathcal{H}$ must vanish.

Model theoretic proof of Theorem 2.

The proof is based on the following version of Hrushovski's **socle theorem**

Proposition 9. (Hrushovski, Pillay-Ziegler)*: let $\mathcal{G}$ be an almost semiabelian D group over K. Let $\mathcal{Y} < \mathcal{G}$ be a coset of $\mathcal{G}^\partial$ and let $\mathcal{Z}$ be an irreducible differential algebraic subset of $\mathcal{Y}$. Let $\mathcal{S} < \mathcal{G}^\partial$ be the stabilizer of $\mathcal{Z}$. Suppose that $\mathcal{S}$ is finite. Then $\mathcal{Z}$ is contained in a coset $\mathcal{Z}'$ of H^∂ where H is the maximal isotrivial D-subgroup of $\mathcal{G}$. Moreover if $\mathcal{Y}, \mathcal{Z}$ are defined over K, so is $\mathcal{Z}'$.*

In order to prove Theorem 2, we consider the set of solutions $\mathcal{Y}$ of $(*)$ as a differential algebraic torsor under the differential algebraic group $\mathcal{G}^\partial$, with everything defined over K. We must show that $\mathcal{Y}$ contains no proper irreducible differential algebraic subvariety $\mathcal{Z}$ over K (indeed, any such $\mathcal{Z}$ is the intersection with $\mathcal{Y}$ of its Zariski closure Z in $\mathcal{G}$.) The proof goes by induction on $dim(\mathcal{G})$.

For a $\mathcal{Z} = Z \cap \mathcal{Y}$ contradicting our claim, let S (resp. $\mathcal{S} = S \cap \mathcal{Y}$) be the stabilizer of Z (resp. $\mathcal{Z}$) in $\mathcal{G}^{\partial}$ (resp.$\mathcal{G}$). Then, S is a proper ADG subgroup of $\mathcal{G}$, and $\mathcal{G}_1 = \mathcal{G}/S$ is an amost semi-abelian D-group. If $dimS > 0$, the induction hypothesis applies; contradiction from the descent of (HX) to $\mathcal{G}_1$. So, $dim(S) = 0$, and by Proposition 9 , $\mathcal{Z}$ is contained in a coset $\mathcal{Z}'$ of H^{∂}, where H is the maximal isotrivial connected D-subgroup of $\mathcal{G}$, all defined over K. Again, we go to the quotient $\pi(\mathcal{G}) = \mathcal{G}/H := \mathcal{G}_1$, where our equation $(*)$ becomes $\partial\ell n_{\mathcal{G}_1}(y_1) = \partial_{L\mathcal{G}_1}x_1$, with $x_1 \in L\mathcal{G}_1(K)$. Moreover, $y_1 = \pi(\mathcal{Z})$ is now a K-rational point of the solution set $\mathcal{Y}_1 = \pi(\mathcal{Y})$. However:

• if $H = \mathcal{G}$, $\mathcal{G}$ descends to $\mathbb{C}$, and our initial solution y contradicts Ax's theorem;
• the induction hypothesis prevents H from being a proper subgroup of $\mathcal{G}$;
• so $\mathcal{G}$ has no non-zero isotrivial subgroup, and must be an extension $\mathcal{A}$ of a traceless abelian variety A by $\mathcal{W} := W_A/U_A$. Since both y_1 and x_1 are K-rational, we conclude by Proposition 5 , as in the Galois proof.