A Course on Motivic Integration

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Talk 1

1. Finite counting / sums
2. Integration

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Still more generally,

$$S_{\psi,\chi,N,\varphi,f,g} := \sum_{x \in \varphi(\mathbb{Z}/N\mathbb{Z})} \psi(f(x))\chi(g(x)),$$

with similar questions.

Recall that

$$S_{\psi,\chi,p,x_1=x_1,x_1,x_1}$$

with p a prime number is a typical Gauss sum over $\mathbb{F}_p$.

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This quest of Motivic Integration tries to be as uniform as possible in number fields as well.

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The above quest of motivic integration also lives here!

- From finite sums to integration on local fields

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Henselian local fields are:

$$\mathbb{F}_q((t))$$

and finite field extensions of

$$\mathbb{Q}_p,$$

the p-adic completion of $\mathbb{Q}$ for the norm $|p^\ell a/b|_p = p^{-\ell}$, $\ell \in \mathbb{Z}$.

$\mathbb{Q}_p$ consists of "Laurent" series of powers of p

$$x = \sum_{i \geq \ell} a_i p^i$$

with $a_i \in \{0, \ldots, p-1\}$,
and ring operations come from approximating x by finite sums

$$\sum_{M \geq i \geq \ell} a_i p^i$$

and calculating in $\mathbb{Q}$.

Summary

compact subrings: the rings of integers
$\mathbb{Z}_p$,
$\mathbb{F}_p[[t]]$,
balls $p^k\mathbb{Z}_p$ around 0, $k \in \mathbb{Z}$,
centered balls:
$a + p^k\mathbb{Z}_p$, disjoint when the a vary.
Translation invariant Haar measure $|dx|$ on $\mathbb{Q}_p$
$p^k\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq p^{-k}\}$,
measure of $\mathbb{Z}_p$ is defined (normalized) as 1, then automatically measure of $a + p^k\mathbb{Z}_p$ is p^{-k}.
This is enough to construct the measure! (See book of Koblitz.)

Summary

On $\mathbb{F}_p((t))$ balls are $a + t^k\mathbb{F}_p[[t]]$,
with measure p^{-k}.

Real valued (Haar) measures, suited to integrate complex valued integrable functions
(Lebesgue theory works. σ-algebra's, approximations, change of variables...)

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ψ can vary, as well as χ_a, a, and $\mathbb{Q}_p$.

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- How does it vary in definable families?

The char p analogue

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Let $\psi : \mathbb{F}_p((t)) \to \mathbb{C}^\times$ be an additive character which is trivial on $p\mathbb{F}_p[[t]]$ and nontrivial on $\mathbb{F}_p[[t]]$.

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New question:
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We will sketch some of these concrete situations, before going to more general axiomatic frameworks.