A Course on Motivic Integration

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Talk 1

1. Finite counting / sums
2. Integration

- Finite counting and finite sums

Let φ be a formula in the language of rings in n free variables.

For each integer $N > 0$ consider the ring

$$\mathbb{Z}/N\mathbb{Z} = \{0, 1, \dots, N-1\}, +, \cdot,$$

and count the number of points in the definable set

$$\varphi(\mathbb{Z}/N\mathbb{Z}) = \{x \in (\mathbb{Z}/N\mathbb{Z})^n \mid \mathbb{Z}/N\mathbb{Z} \models \varphi(x)\}.$$

- How vary these numbers $\sharp\varphi(\mathbb{Z}/N\mathbb{Z})$ with $N > 0$?
- How do they vary in definable families?

For each $N > 0$ and for each group homomorphism

$$\psi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^\times$$

to the complex unit circle,

consider the sum

$$S_{\psi,N,\varphi,f} := \sum_{x \in \varphi(\mathbb{Z}/N\mathbb{Z})} \psi(f(x)),$$

with f a definable function in the ring language.

How do these finite sums $S_{\psi,N,\varphi,f}$ vary with $N > 0$,

and how do they vary in definable families?

For each $N > 0$ and for each group homomorphism

$$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$$

to the complex unit circle, considering similar sums $S_{\chi,N,\varphi,f}$, one can raise the same questions.

Still more generally,

$$S_{\psi,\chi,N,\varphi,f,g} := \sum_{x\in\varphi(\mathbb{Z}/N\mathbb{Z})} \psi(f(x))\chi(g(x)),$$

with similar questions.

Recall that

$$S_{\psi,\chi,p,x_1=x_1,x_1,x_1}$$

with p a prime number is a typical Gauss sum over $\mathbb{F}_p$.

One of the quests of Motivic Integration can be described as the search for a as wide as possible class of formulas φ and definable functions f, g, so that for as many as possible numbers $N > 0$ and choices of the characters φ, χ, the dependence of $S_{\psi,\chi,N,\varphi,f,g}$ on N (or of similar even more general objects) (and the dependence in definable families) can be understood in terms of as few as possible

"basic objects" of a similar kind $S_{\psi,\chi,N,\varphi,f,g}$,

where "basic" often means that N is only allowed to be a prime number,

or even better: in terms of an abstract, "basic" geometric object.

Of course one can start with any ring of integers $\mathcal{O}$ in any number field (instead of with $\mathbb{Z}$)

and consider finite quotients $\mathcal{O}/I$ instead of $\mathbb{Z}/N\mathbb{Z}$.

This quest of Motivic Integration tries to be as uniform as possible in number fields as well.

- A motivation for focusing on $N = p^\ell$ with p prime

By cheating a bit (that is, hiding some difficulties),

one can say that the most relevant case is to consider $N = p^\ell$ for all primes p and $\ell > 0$,

namely by the Chinese Remainder Theorem

$$\mathbb{Z}/12\mathbb{Z} \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}.$$

If φ is quantifier free and positive and f is a polynomial, then easy up to prime factorization of N.

For more complicated φ, ...

Convention: From now on, we only consider $N = p^\ell$ with p prime (more generally, powers of prime ideals in rings of integers).

- A char p analogue

Instead of working on $\mathbb{Z}/p^\ell\mathbb{Z}$,

we can also work with the finite rings

$$\mathbb{F}_p[t]/(t^\ell)$$

and study similar finite counting & summation problems uniformly in p and ℓ.

The above quest of motivic integration also lives here!

- From finite sums to integration on local fields

Henselian local fields are:

$$\mathbb{F}_q((t))$$

and finite field extensions of

$$\mathbb{Q}_p,$$

the p-adic completion of $\mathbb{Q}$ for the norm $|p^\ell a/b|_p = p^{-\ell}$, $\ell \in \mathbb{Z}$.

$\mathbb{Q}_p$ consists of "Laurent" series of powers of p

$$x = \sum_{i \geq \ell} a_i p^i$$

with $a_i \in \{0, \dots, p-1\}$,
and ring operations come from approximating x by finite sums

$$\sum_{M \geq i \geq \ell} a_i p^i$$

and calculating in $\mathbb{Q}$.

Summary

compact subrings: the rings of integers
$\mathbb{Z}_p$,
$\mathbb{F}_p[[t]]$,
balls $p^k\mathbb{Z}_p$ around 0, $k \in \mathbb{Z}$,
centered balls:
$a + p^k\mathbb{Z}_p$, disjoint when the a vary.
Translation invariant Haar measure $|dx|$ on $\mathbb{Q}_p$
$p^k\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq p^{-k}\}$,
measure of $\mathbb{Z}_p$ is defined (normalized) as 1, then automatically measure of $a + p^k\mathbb{Z}_p$ is p^{-k}.
This is enough to construct the measure! (See book of Koblitz.)

Summary

On $\mathbb{F}_p((t))$ balls are $a + t^k\mathbb{F}_p[[t]]$,
with measure p^{-k}.

Real valued (Haar) measures, suited to integrate complex valued integrable functions
(Lebesgue theory works. σ-algebra's, approximations, change of variables...)

Back to (definable) integration

Let $\psi : \mathbb{Q}_p \to \mathbb{C}^\times$ be an additive character

which is trivial on $p\mathbb{Z}_p$ and nontrivial on $\mathbb{Z}_p$.

For $a \in \mathbb{Z}_p$ let $\chi_a : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ be a multiplicative character to the complex unit circle,

which is trivial on $1 + pa\mathbb{Z}_p$ and nontrivial on $1 + a\mathbb{Z}_p$.

Let φ be a formula in n free variables, in any language of valued fields (out of a bunch of natural languages for valued fields)

and let f, g be definable functions.

Then, for any such data, one can consider the integral

$$I_{\psi,\chi,\mathbb{Q}_p,\varphi,f,g}(a) := \int_{\{x\in\mathbb{Q}_p^n|\varphi(x)\}} \psi(f(x))\chi_a(g(x))|dx|,$$

if it is absolutely integrable.

ψ can vary, as well as χ_a, a, and $\mathbb{Q}_p$.

$$I_{\psi,\chi,\mathbb{Q}_p,\varphi,f,g,h}(a) := \int_{\{x\in\mathbb{Q}_p^n \mid \varphi(x)\}} \psi(f(x))\chi_a(g(x))|dx|,$$

- How does $I_{(\cdot)}$ depend on ψ, χ_a, and a?
- More importantly, how does it depend on $\mathbb{Q}_p$?
- How does it vary in definable families?

The char p analogue

Likewise,
Let $\psi : \mathbb{F}_p((t)) \to \mathbb{C}^\times$ be an additive character which is trivial on $p\mathbb{F}_p[[t]]$ and nontrivial on $\mathbb{F}_p[[t]]$.

For $a \in \mathbb{F}_p[[t]]$ let $\chi_a : \mathbb{F}_p((t))^\times \to \mathbb{C}^\times$ be a multiplicative character to the complex unit circle,
which is trivial on $1 + ta\mathbb{F}_p[[t]]$ and nontrivial on $1 + a\mathbb{F}_p[[t]]$.

Then, for any such data, one can consider the integral

$$I_{\psi,\chi,\mathbb{F}_p((t)),\varphi,f,g,h}(a) := \int_{\{x \in \mathbb{F}_p((t))^n \mid \varphi(x)\}} \psi(f(x))\chi_a(g(x))|dx|,$$

if it is absolutely integrable.

Same questions:
how does it depend on $\mathbb{F}_p((t))$ and how does it vary in definable families?

New question:
How does $I_{(\cdot)}$ depend on the characteristic?

The quest of Motivic Integration can be re-described as the search for a as wide as possible class of formulas φ and definable functions f, g, so that for as many as possible local fields K choices of the characters φ, χ, the dependence of $I_{\psi,\chi,K,\varphi,f,g}$ on the data (or of similar even more general objects) (and the dependence in definable families) can be understood in terms of as few as possible "basic objects" of the kind $S_{\psi,\chi,\mathbb{F}_q,\varphi,f,g}$,

with $S_{(\cdot)}$ as before, and $\mathbb{F}_q$ the residue field of K,

or even better: in terms of abstract, "basic" geometric objects.

Throughout the short history of motivic integration,

while focusing on different classes of φ, f, g,

on different collections of Henselian local fields K (in the beginning, for residual characteristic big enough),

with or without characters ψ and χ_a,

and with or without parameter dependence (i.e. in definable families),

many different constructions were used,

as well as many different kinds of "basic" geometric objects.

We will sketch some of these concrete situations, before going to more general axiomatic frameworks.