A Course on Motivic Integration, II

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Talk 2

1. Geometric Motivic Integration: Kontsevich, Denef - Loeser
2. Arithmetic Motivic Integration: Denef - Loeser

- Kontsevich, Denef - Loeser approach of geometric motivic integration

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Major advantage:

- A geometric object is used to interpolate the local integrals, which moreover contains many kinds of geometric data, apart from the interpolating power! (Kontsevich)

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In the completed localisation of $K_0(\mathrm{Var})$, the approximation is converging to the "measure of $\varphi(\cdot((t)))$".

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on the applications one has in mind or even already on the theory of motivic integration one is looking at (geometric / arithmetic / Cluckers - Loeser / Hrushovski - Kazhdan / b-minimal,...)

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Recall from talk 1: basic objects for understanding the integrals $I_{(\cdot)}$ over local fields $\mathbb{Q}_p$ resemble finite sums $S_{(\cdot)}$ over some definable set in the residue field $\mathbb{F}_p$ (uniformly in $\mathbb{Q}_p$ and $\mathbb{F}_p$).

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Of course, an (affine) variety X over $\mathbb{Q}$ determines such sums (here just finite counting instead of finite sums) by taking $\sharp X(\mathbb{F}_p)$ for p big.

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Thus, the notion "p big" here depends on the choice of equations f_i... but this is a standard, trivial issue in algebraic geometry.

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$$K_0(\mathrm{Var})[1/\mathbb{L}].$$

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p-adically, one can approximate sets by families of balls $a + p^k\mathbb{Z}_p$ as long as the balls in the family yield a converging sum of their measures. So, a small ball is a ball $a + p^k\mathbb{Z}_p$ with big $k >> 0$ and then the measure is p^{-k}.

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This is a filtration and one completes w.r.t. this filtration.

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- where it is clear in both case what the uniformity in k means.

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  - The equivalence class of formulas to which one associates measures is not really natural if one intends to interpolate p-adic and $\mathbb{F}_p((t))$ - integrals, see $(^\circ)$ and $(^{\circ\circ})$.

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namely, in a similar completion of a similar localisation.
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Moreover, other "basic" geometric objects arise by application of a ring homomorphism
from $K_0(\mathrm{Th}_{pf})$ to a ring over $K_0(\text{Chow Motives}) \otimes \mathbb{Q}$.

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In talk 3 more theories of motivic integration!