A Course on Motivic Integration, II

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Talk 2

1. Geometric Motivic Integration: Kontsevich, Denef - Loeser

2. Arithmetic Motivic Integration: Denef - Loeser

- Kontsevich, Denef - Loeser approach of geometric motivic integration

One takes:

- φ quantifier free (in the language of Denef - Pas),
- no characters χ_a, neither ψ,
- "basic" geometric objects: in a ring over $K_0(\mathrm{Var})$, namely a completion of a localisation.

In terms of these basic objects, the theory understands the integrals

$$I_{\varphi,K} := \int_{\{x\in K^n \mid \varphi(x)\}} |dx| = \mu(\varphi(K)),$$

when K varies over Henselian local fields of big enough residual characteristic.

Moreover, for such K,
$\mu(\varphi(K))$ only depends on the residue field of K. Hence, for isomorphic residue fields, one finds the same measure!

Drawbacks:

- No parameter dependence in definable families
- The "completion" is used (and how much information is lost in the completion process?)
- No quantifiers in φ, no characters in the integrand, thus only "basic" kinds of integrals over local fields can be interpolated in this theory.

Major advantage:

- A geometric object is used to interpolate the local integrals, which moreover contains many kinds of geometric data, apart from the interpolating power! (Kontsevich)

Some details

At first, one interprets φ in fields $k((t))$, with k a field of char 0, for any such k.
Secondly, one "approximates" $\varphi\big(k((t))\big)$ by "cylinders", uniformly in k.
One defines the measure of cylinders, uniformly in k.

In the completed localisation of $K_0(\mathrm{Var})$, the approximation is converging to the "measure of $\varphi(\cdot((t)))$".

($^\circ$) To what is one associating a measure here?

One can answer this question ($^\circ$) in many slightly different ways, but, in a certain sense,

one associates the measure to the functor sending k to $\varphi\big(k((t))\big)$.

in another sense one associates the measure to a certain equivalence class of formulas, namely equivalent to φ in some way.

The equivalence relations on formulas φ which yield the same definable set for "some" collection of suitable models depends heavily

on the applications one has in mind or even already on the theory of motivic integration one is looking at (geometric / arithmetic / Cluckers - Loeser / Hrushovski - Kazhdan / b-minimal,...)

$K_0(\mathrm{Var})$

$K_0(\mathrm{Var})$ is the abelian group generated by isomorphism classes of varieties over $\mathbb{Q}$,
divided out by the relation

$$[X] = [X \setminus Y] + [Y]$$

for Y a closed subvariety of X.

Recall from talk 1: basic objects for understanding the integrals $I_{(\cdot)}$ over local fields $\mathbb{Q}_p$ resemble finite sums $S_{(\cdot)}$ over some definable set in the residue field $\mathbb{F}_p$ (uniformly in $\mathbb{Q}_p$ and $\mathbb{F}_p$).
Of course, an (affine) variety X over $\mathbb{Q}$ determines such sums (here just finite counting instead of finite sums) by taking $\sharp X(\mathbb{F}_p)$ for p big.

($^{\circ\circ}$) On what is one counting here, on $X(\mathbb{F}_p)$?

This issue ($^{\circ\circ}$) is related to issue ($^{\circ}$). One does not literally look at $X(\mathbb{F}_p)$, because this is not defined (X is over $\mathbb{Q}$). On the other hand, for (affine) X one can look at a finite collection of polynomials f_i that define X in $\mathbb{A}^n$, and these polynomials live in some ring $\mathbb{Z}[1/N]$ for some $N > 0$.
So X can naturally be replaced by X_0, the (affine) variety over $\mathbb{Z}[1/N]$ defined similarly by the f_i.
Hence, for $p > N$ one can look at the set $X_0(\mathbb{F}_p)$.

Thus, the notion "p big" here depends on the choice of equations f_i... but this is a standard, trivial issue in algebraic geometry.

The localisation

Write $[\mathbb{A}^1]$ as $\mathbb{L}$. Then, as in the foregoing, $\mathbb{L}$ corresponds, by writing $\sharp(\mathbb{A}^1(\mathbb{F}_p)) = \sharp\mathbb{F}_p = p$, to p, for p big.
Recall that $\mu(p^k\mathbb{Z}_p) = p^{-k}$ for $k \in \mathbb{Z}$, and thus,
it is natural and necessary, in order to interpolate p-adic integrals, to have access to p^{-k} at some motivic level.

The most natural localisation that does this :

$$K_0(\mathrm{Var})[1/\mathbb{L}].$$

The completion

p-adically, one can approximate sets by families of balls $a + p^k\mathbb{Z}_p$ as long as the balls in the family yield a converging sum of their measures. So, a small ball is a ball $a + p^k\mathbb{Z}_p$ with big $k >> 0$ and then the measure is p^{-k}. Hence, in any sense, p^{-k} is considered as small for $k >> 0$.
By a certain coincidence, this corresponds to the natural metric on the real numbers, so any converging sum of terms p^{-k} yields a real number, since $\mathbb{R}$ is complete.

On the other hand, we can say that $\mathbb{L}^{-k}$ is small for $k >> 0$, (since it corresponds to p^{-k}), but there is no reason why any infinite sum of small powers of $\mathbb{L}$ should converge in $K_0(\mathrm{Var})[1/\mathbb{L}]$. In particular, there is no metric on $K_0(\mathrm{Var})[1/\mathbb{L}]$.

The completion

The completion: $[X]/\mathbb{L}^k$ is considered small when k is big and the dimension of X is small compared to k.

(Note that in this case $\sharp X(\mathbb{F}_p)/p^k$ is small for p big enough, so we are considering small "real numbers" here.)

Formally, for each $m > 0$ let F_m be the subgroup of $K_0(\mathrm{Var})[1/\mathbb{L}]$ generated by quotients $[X]/\mathbb{L}^k$ with $\dim X - \dim \mathbb{L}^k \leq -m$. (Of course $\dim \mathbb{L}^k = k$.)
This is a filtration and one completes w.r.t. this filtration.

Cylinders

A cylinder is a subset $C(k)$ of $k[[t]]^n$, "uniform in k in some sense", such that
for a projection

$$\pi_m : k[[t]]^n \to (k[t]/t^m)^n, \text{ uniformly in } k,$$

one has that

- $\pi_m(C(\cdot))$ is a constructible subset of the corresponding affine space $\mathbb{A}^{nm}$
- $C(\cdot) = \pi_m^{-1}(\pi_m(C(\cdot)))$,
- where it is clear in both case what the uniformity in k means.

- Technical drawbacks:
  - Not every cylinder is definable in the original Denef - Pas language!

Suggested solution (suggested by Denef) to make this aspect of the theory more smooth:
one could rewrite the theory using angular components of higher order (modulo t^m), which were not available at that time.

- The equivalence class of formulas to which one associates measures is not really natural if one intends to interpolate p-adic and $\mathbb{F}_p((t))$ - integrals, see $(^\circ)$ and $(^{\circ\circ})$.

Flexibility of the theory

- φ may have coefficients (parameters) from a (base) ground field k_0 in which case one only considers fields k over k_0 and one replaces Var (the varieties over $\mathbb{Q}$) by Var_{k_0} (varieties over k_0).
- In some sense, parameter dependence of integer parameters, in a (still quantifier free) definable family, can be understood and interpolates the corresponding parameterized integrals over local fields.

- Denef - Loeser approach of arithmetic motivic integration

φ with quantifiers, still in the language of Denef - Pas.
no characters χ_a, neither ψ.
basic geometric objects: in a ring over $K_0(\mathrm{Th}_{pf})$, with Th_{pf} the theory of pseudofinite fields,
namely, in a similar completion of a similar localisation.
Uses QE (quantifier elimination) in valued fields, QE in Th_{pf}, QE for $\mathbb{Z}$.

Moreover, other "basic" geometric objects arise by application of a ring homomorphism
from $K_0(\mathrm{Th}_{pf})$ to a ring over $K_0(\text{Chow Motives}) \otimes \mathbb{Q}$.

In terms of these basic objects, the theory understands the integrals

$$I_{\varphi,K} := \int_{\{x \in K^n \mid \varphi(x)\}} |dx| = \mu(\varphi(K)),$$

when K varies over Henselian local fields of big enough residual characteristic.

Moreover, for such K,

$\mu(\varphi(K))$ only depends on the residue field of K. Hence, for isomorphic residue fields, one again finds the same measure!

This is a kind of transfer principle that generalizes the Ax-Kochen Ershov principle from the truth-value of sentences (which is in a sense just the discrete measure of either the empty set or a one point set) to the measure of definable sets, defined by formulas in certain languages.

It focuses in both cases on the same fields, namely nonarchimedean local fields with isomorphic residue field and big enough residue characteristic (in relation to φ).

Drawbacks:

- No general parameter dependence in definable families
- The "completion" is used (and how much information is lost?)
- No characters in the integrand
- Still not every cylinder is definable! (The same reason and the same suggestion apply.) And issues (°), (°°) remain subtle.

Advantages:

- A geometric object is used to interpolate the local integrals, which again contains many kinds of geometric data (namely all geometric data contained in some ring over $K_0(\mathrm{Th}_{pf})$ or over $K_0(\text{Chow Motives})$).
- Again, parameter dependence (in definable families) of integer variables is understood. This feature yields motivic versions of many kinds of Denef style rationality results.

In talk 3 more theories of motivic integration!