A Course on Motivic Integration, III

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Model theory and Applications
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Talk 3

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We will again try to convey general ideas instead of the full formalisms.

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Moreover, the previous theories of motivic integration can be recovered by applying ring homomorphisms from $K_0(\mathrm{Th}_{Fields})$ to $K_0(\mathrm{Th}_{pf})$ or to $K_0(\mathrm{Var})$.
Issue ($^{\circ\circ\circ}$): A slightly different K_0 is used if ψ is involved, about which later more.

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and thus, a naive version of the transfer principle makes no sense.

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holds, only depends on the residue field of K.

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Or, the arguments f_1, f_2 of ψ are really different (that is, have great p-adic distance). In this case play with the choices of ψ to talk the terms apart. Hence this case can not occur! □

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The application to several variants of the Fundamental Lemma has been worked out recently (see C. - Hales - Loeser) and to aspects of representation theory by Gordon and Cunningham.

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Certainly we still need a motivic object for $p^{-\ell}$ for $\ell \in \mathbb{Z}$, and thus $K_0(\mathrm{Th}_{Fields})[1/\mathbb{L}]$, but we will need to further localise (instead of to complete). This necessity can be seen as follows.

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Hence clearly one needs to have access to fractions $\frac{1}{1-p^{-\ell}}$ for $\ell > 0$ in a motivic way, which is most easily done by taking

$$K_0(\mathrm{Th}_{Fields})[1/\mathbb{L}, \frac{1}{1-\mathbb{L}^{-\ell}}]_{\ell>0}.$$

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which (in our framework) just equals the corresponding derivative of the rational function

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- A solution is offered by allowing the language to be as general as possible (e.g. just requiring *b*-minimality).

A heuristic

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However, for the moment, on the reals there is only a natural theory of integration (w.r.t. the Lebesgue measure) on the subanalytic structure! (see Lion - Rolin - Comte)

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by applying QE for algebraically closed fields to send a formula in the language of rings to a constructible set,
resp. by applying QE for pseudo-finite fields similarly.

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Thus, the theory recovers the geometric and arithmetic integration theories.

(Note that the arithmetic theory does not recover the geometric theory, nor vice versa.)

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where isomorphism means definable bijection leaving the variables running over $\varphi(\cdot)$ unchanged.

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One builds the relative $(K_0^{\mathrm{exp}})_Z$ in a similar way.

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- advantages of the theory:
  Geometric objects are used that can specialize to all previously used geometric objects, as well as to integrals over local fields. (However, no transfer principle is worked out, but one can plug in, at some point, Cluckers - Loeser's form of the transfer principle.)

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p-adic interpolation only for residual char $>> 0$,

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Hence, an integral is replaced by a "sum" or a "geometric basic object" over the auxiliary sorts "value group" and "'residue field", or over RV, the natural combination of the two.

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That is, one (technically) just needs a definable function from the auxiliary sorts to the valued field which approximates good enough the balls, put otherwise, which can play the role of constant term in the argument of ψ.