A Course on Motivic Integration, III

Raf Cluckers

École normale supérieure, Paris
KULeuven, Belgium

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Talk 3

In the Cluckers - Loeser approach, many credits go to Denef, who paved the way by his many results on p-adic integrals.

We will again try to convey general ideas instead of the full formalisms.

- Cluckers - Loeser approach of motivic integration

φ with quantifiers, definable in a language of Denef - Pas,
any additive character ψ as in talk 1 can be used (uniform in the choice of ψ)
f a definable function (serving as the argument of ψ),
no characters χ_a.

Basic geometric objects: in a localisation of $K_0(\mathrm{Th}_{Fields})$, with Th_{Fields} the theory of fields of characteristic zero,
Uses QE in valued fields, QE for $\mathbb{Z}$, cell decomposition.

Moreover, the previous theories of motivic integration can be recovered by applying ring homomorphisms from $K_0(\mathrm{Th}_{Fields})$ to $K_0(\mathrm{Th}_{pf})$ or to $K_0(\mathrm{Var})$.
Issue $(^{\circ\circ\circ})$: A slightly different K_0 is used if ψ is involved, about which later more.

Note that no completion of K_0 is needed anymore!

Instead one uses an adapted localisation of K_0 (which is needed to interpolate p-adic integrals and which is sufficient).

In terms of these basic objects or of ($^{\circ\circ\circ}$), the theory understands the integrals

$$I_{\varphi,K} := \int_{\{x\in K^n|\varphi(x)\}} \psi(f)|dx|,$$

when K varies over Henselian local fields of big enough residual characteristic.
And similarly in definable families using any kinds and any number of parameters and quantifiers!

However, by the very nature of ψ, which has a different range (image) when the local field varies (not only depending on the residue field), this integral does not only depend on the residue field,
and thus, a naive version of the transfer principle makes no sense.

The Cluckers - Loeser generalisation of the transfer principle

Theorem (Rough version)

Let φ_i, f_i be Denef - Pas definable for $i = 1, 2$ (possibly living in a definable family, that is, depending on parameters).
Then for any Henselian local field K with big enough residual characteristic,
whether for each choice of ψ the equality

$$I_{\psi,K,\varphi_1,f_1} = I_{\psi,K,\varphi_2,f_2}$$

holds, only depends on the residue field of K.

The Cluckers - Loeser generalisation of the transfer principle

Very rough proof.

First reduce to "small" φ_i by calculating the integrals uniformly to some extend, that is, express them in terms of the residue field as far as possible. In other words, pull them back to sums over auxiliary sorts as far as possible.
Then, either f_1 and f_2 are p-adically close and ψ plays the same role in the left and in the right hand side, so, there is no real ψ and one can easily apply the classical transfer principle.
Or, the arguments f_1, f_2 of ψ are really different (that is, have great p-adic distance). In this case play with the choices of ψ to talk the terms apart. Hence this case can not occur! □

The Cluckers - Loeser generalisation of the transfer principle

This transfer principle (with parameter dependence and with ψ) is very promising for applications in the Langlands program and representation theory.
The application to several variants of the Fundamental Lemma has been worked out recently (see C. - Hales - Loeser) and to aspects of representation theory by Gordon and Cunningham.

The localisation of K_0

$K_0(\mathrm{Th}_{Fields})$ is the abelian group generated by isomorphism classes of formulas in the ring language,
where isomorphisms are definable bijections for Th_{Fields},
divided out by the relation

$$[X] = [X \setminus Y] + [Y]$$

for Y a definable subset of X.

Certainly we still need a motivic object for $p^{-\ell}$ for $\ell \in \mathbb{Z}$, and thus $K_0(\mathrm{Th}_{Fields})[1/\mathbb{L}]$, but we will need to further localise (instead of to complete). This necessity can be seen as follows.

The localisation of K_0

? What is the measure of the set $A \subset \mathbb{Z}_p$ of p-adic integers with even order and first p-adic digit equal to 1? (An easily definable set.)
Certainly $A = \bigcup_{i\in\mathbb{N}} \; p^{2i} + p^{2i+1}\mathbb{Z}_p$, a disjoint union of balls.
Since $\mu(p^{2i} + p^{2i+1}\mathbb{Z}_p) = \mu(p^{2i+1}\mathbb{Z}_p) = p^{-2i-1}$ (the measure of a ball),
One calculates $\sum_{i\in\mathbb{N}} p^{-2i-1} = p^{-1}\sum_{i\in\mathbb{N}}(p^{-2})^i = p^{-1}\frac{1}{1-p^{-2}}$ (a geometric sum).
Hence clearly one needs to have access to fractions $\frac{1}{1-p^{-\ell}}$ for $\ell > 0$ in a motivic way, which is most easily done by taking

$$K_0(\mathrm{Th}_{Fields})[1/\mathbb{L}, \frac{1}{1-\mathbb{L}^{-\ell}}]_{\ell>0}.$$

The localisation of K_0

Note that, in a "dimensional" completion of $K_0(\mathrm{Th}_{Fields})[1/\mathbb{L}]$, one has already motivic access to $\frac{1}{1-\mathbb{L}^{-\ell}}$ since in such a completed ring

$$\frac{1}{1-\mathbb{L}^{-\ell}} = \sum_{i\in\mathbb{N}} \mathbb{L}^{-\ell i},$$

with on the right hand side a converging sum.

Hence, the new localisation is not only necessary, it is also more natural than the completion, in the sense that no information is lost any more in the completion process.

The localisation of K_0

Why does this localisation work to measure "definable" sets uniformly/motivically (and also in definable families)?
$\rightarrow$ The only integer dependence that can possibly play is piecewise linear in nature,
thus, the only infinite sums parameterized by definable subsets of $\mathbb{Z}^n$ that occur, are geometric series or its derivatives, like

$$\sum_{i \in \mathbb{N}} (i+1)\mathbb{L}^{-i}$$

which (in our framework) just equals the corresponding derivative of the rational function

$$\frac{1}{1-\mathbb{L}^{-1}} \text{ in } \mathbb{L}.$$

Technical drawbacks

(1) In some sense, the localisation has been "chosen" and is not natural, although it is the most evident localisation to yield the numbers $p^{-\ell}$ and $\frac{1}{1-p^{-\ell}}$ uniformly in p.

- A solution is offered in the Hrushovski - Kazhdan approach, where essentially no localisation at all is used!

- Another solution might be to show that this localisation is natural, that is, a necessary choice. This is open problem ($^{\circ\circ\circ\circ}$).

Technical drawbacks

(2) One can no longer measure in "σ-algebra's", since an infinite Boolean combination of definable sets is not always a definable set. So the collections of measurable sets is confined here to definable sets in some languages.

- A solution is offered by allowing the language to be as general as possible (e.g. just requiring *b*-minimality).

A heuristic for why varying the language in a broad class of general languages is sufficiently interesting:
Every single concrete integral that one encounters seems to be built up with definable data in some natural language!! (also over archimedean local fields!)

Compare it to the richness of the collection of *o*-minimal theories. In any fixed *o*-minimal theory, you are not allowed to take infinite Boolean combinations, but many interesting sets in $\mathbb{R}^n$ live inside some *o*-minimal (or more generally *d*-minimal) structure!

However, for the moment, on the reals there is only a natural theory of integration (w.r.t. the Lebesgue measure) on the subanalytic structure! (see Lion - Rolin - Comte)

The ring homomorphisms to the classical K_0

The ring homomorphism from

$$K_0(\mathrm{Th}_{Fields})[1/\mathbb{L}, \frac{1}{1-\mathbb{L}^{-\ell}}]_{\ell>0}$$

to the completion of the localisation of $K_0(\mathrm{Var})$, resp. of $K_0(\mathrm{Th}_{pf})$, is naturally induced by

$$K_0(\mathrm{Th}_{Fields}) \to K_0(\mathrm{Var})$$

$$\text{resp. } K_0(\mathrm{Th}_{Fields}) \to K_0(\mathrm{Th}_{pf}),$$

by applying QE for algebraically closed fields to send a formula in the language of rings to a constructible set,
resp. by applying QE for pseudo-finite fields similarly.

The ring homomorphisms to the classical K_0

This way, disjoint unions are sent to disjoint unions, and Cartesian products to Cartesian products.
In any case, $1/\mathbb{L}$ is sent to $1/\mathbb{L}$ in the localisations and as mentioned before, $\frac{1}{1-\mathbb{L}^{-\ell}}$ also makes sense in the completed rings.

Thus, the theory recovers the geometric and arithmetic integration theories.

(Note that the arithmetic theory does not recover the geometric theory, nor vice versa.)

The relative K_0,

The relative K_0, that is, with parameter dependence

For any formula φ of valued fields in any kinds of variables, thus yielding a functor Z which sends a field k to $\varphi\big(k((t))\big)$, one considers the relative Grothendieck ring over Z (or with parameters from Z):

$$K_0(\mathrm{Th}_{Fields})_Z$$

(and its corresponding localisation), as generated by the isomorphism classes of all definable subsets X (in the considered language) of

$$k^n \times \varphi\big(k((t))\big), \text{ uniformly in } k$$

where isomorphism means definable bijection leaving the variables running over $\varphi(\cdot)$ unchanged.

The K_0 used for ψ, see issue $(^{\circ\circ\circ})$

K_0^{exp} is the abelian group generated by isomorphism classes of triples (φ, f, ξ),
with φ a formula in the ring language,
$f : \varphi(k) \subset k^n \to k((t))$ definable in a Denef - Pas language, uniformly in k
and $\xi : \varphi(k) \to k$ similarly definable,
where isomorphisms are definable bijections for Th_{Fields}, making commuting diagrams with the functions f and ξ,
divided out by some natural relations
inspired by the intended p-adic meaning of (φ, f, ξ) as
$S_{\varphi,p,\psi,f+\xi} = \sum_{x\in\varphi(\mathbb{F}_p)} \psi(f(x) + \xi(x))$

The K_0 used for ψ, see issue $(^{\circ\circ\circ})$

Recall that ψ is supposed to be trivial on the maximal ideal and nontrivial on the valuation ring, so that

$$\psi(f(x) + \xi(x))$$

makes sense.

One builds the relative $(K_0^{\exp})_Z$ in a similar way.

Nonnegativity for K_0: the semi-group

Instead of looking at the groups generated by isomorphisms classes of ..., it is technically better to look at Grothendieck semi-groups, denoted SK_0 or K_0^+, generated by isomorphism classes divided out by e.g. disjoint union relations.

This is done in order to speak of nonnegative functions, or to bound a function by another (integrable) function.

Another advantage is that SK_0 contains most often more information than K_0, e.g., the Grothendieck ring of definable subsets of $\mathbb{Q}_p^n$ is trivial, but not so for the semi-ring (which contains $\mathbb{N}$)!

An analytic language

The theory is also worked out (yet only to some extend) for enriched Denef - Pas languages by function symbols for a collection of analytic functions, and which make sense uniformly in nonarchimedean local fields (see C - Lipshitz - Robinson).

A key point is to restrict the rich analytic languages introduced by Lipshitz to a smaller language that makes sense on local fields. This key point was first made by van den Dries by considering T-adically strictly converging power series over $\mathbb{Z}[[T]]$.

Drawbacks:

- Not yet characters χ_a uniformly in a,
- in any model one chooses an angular component map, or a uniformizer, which is an unnatural choice, although the theory works uniformly in this choice.
- Issues $(^\circ)$ and $(^{\circ\circ})$ remain subtle.

Advantages:

- Again a geometric object is used to interpolate the local integrals, which moreover specializes to the previously used geometric objects.
- Parameter dependence (in definable families), the characters ψ, the transfer principle.

- Hrushovski - Kazhdan approach approach to motivic integration

φ with quantifiers, definable in a language of Denef - Pas, or more naturally, in the RV-language,
any additive character ψ as in talk 1 can be used (uniform in the choice of ψ)
f a definable function (serving as the argument of ψ),
no characters χ_a.

Recall: For any valued field K with maximal ideal M_K of its valuation ring, $RV(K) := K^\times/(1+M_K)$, the quotient of multiplicative groups, which sees at the same time the residue field and the value group, hence the name RV.
basic geometric objects: in rings (or again semi-rings) similar to K_0(definable subsets of RV^n), thus one does not sum up over the value group, one just keeps the definable sets also at the value group level. One uses QE in alg. closed valued fields.

- The Hrushovski - Kazhdan approach is slightly more natural by the use of RV instead of an angular component. Algorithmically there is no apparent advantage (the same steps need to be done to interpolate p-adic integrals, possibly in some different order, in both the C-L and the H-K approaches).
- advantages of the theory:
  Geometric objects are used that can specialize to all previously used geometric objects, as well as to integrals over local fields. (However, no transfer principle is worked out, but one can plug in, at some point, Cluckers - Loeser's form of the transfer principle.)

Major breakthrough of H - K approach

(*) Abstract, surprising, isomorphisms of Grothendieck (semi-) rings are established.

? What does motivic integration try to do?

- Possible answer: Attach invariants to (equivalence classes of) formulas for valued fields (for some equivalence relation, e.g. yielding the same set for a certain collection of models). These invariants are intended to be as general as possible.
- By these isomorphisms (*), one shows that one gets hold of the most general possible invariants, namely all that rests after fixing some equivalence relations.

This is done in their work for several kinds of equivalence relations, e.g. related to definable sets with isomorphisms respecting the Jacobian, and so on.

Drawback:

- no analytic theory is (by now) included which makes sense on local fields (only some analytic theory on $\mathbb{C}_p$ for $p >> 0$ is included), but this is probably just a matter of time to get it in.

- Common features (C - L, H - K)

A motivic version of change of variables formulae along definable bijections,
also in definable families (that is, with parameter dependence).

- common drawbacks of all considered theories:

p-adic interpolation only for residual char $>> 0$,

no characters χ_a uniformly in a,

subtlety of issues $(^\circ)$, $(^{\circ\circ})$ remains.

What makes these parameter dependences doable in these C-L, H-K approaches?

In both theories, one starts with integrals in one variable (as Denef did with his cell decomposition) (possibly with some other parameters), and, in one variable, the domain is an easy combination of balls and a measure zero set, such that on these balls the integrand is constant.
Hence the integral is just the measure of the ball times the value of the integrand on it, summed up over all the occurring balls (which are usually parameterized by subsets of the integers and the residue field).
Hence, an integral is replaced by a "sum" or a "geometric basic object" over the auxiliary sorts "value group" and "'residue field", or over RV, the natural combination of the two.

What makes ψ controllable in these C-L, H-K approaches?

Common idea: replace the argument of ψ by a linear function after applying a suitable change of variables.
Then write the domain as a simple combination of balls and a measure zero set.
On such a collection of balls and a measure zero set, the linear function which is the argument of ψ is easy to calculate.
That is, one (technically) just needs a definable function from the auxiliary sorts to the valued field which approximates good enough the balls, put otherwise, which can play the role of constant term in the argument of ψ.