A Course on Motivic Integration IV

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Talk 4

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Again we try to transmit as much intuition and goals as possible.

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b-minimality has left the realm of valued fields and became an abstract theory (which is in particular still useful for valued fields), consisting of three axioms (b1), (b2), (b3), where (b1) remains the essential one in most situations, and where (b2) and (b3) are included to have a good dimension theory in full generality.

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A theory is a (b1)-theory if and only if all its models are.

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By the example "Henselian valued fields of characteristic 0" this allows for all p-adic fields to be models hence removes the issues $(^\circ)$ and $(^{\circ\circ})$, that is, no ambiguity of working with $k((t))$ with k of char 0 when one is actually interested in p-adic fields.

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However, it is not clear how to define an interesting, purely geometric object that interpolates these integrals in the generality of b-minimality, other than the "geometric" object that merely retains finite counting (which one can conjecture implies having many geometric data as well...).

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Lemma

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Lemma

The dimension of X is well defined and satisfies all desirable properties.

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small p. Stil better: the Iwasawa logarithm.

- the χ_a and ψ at the same time.
- Issue $(^{\circ\circ\circ\circ})$: is the localisation used by C - Loeser canonical, and if so, is there an isomorphism in the C - Loeser framework resembling the isomorphism of Hrushovski - Kashdan?

Description of references

The possibility to apply model theoretical tools to study *p*-adic integrals was opened up by Macintyre's QE [18] and Cohen's [9] work on cell decomposition. This did the job for definable *p*-adic integrals for Denef [10] and in [11] he reproves Macintyre's QE and cell decomposition à la Cohen. Denef - Pas and Macintyre treat uniformity issues for cell decomposition and *p*-adic integrals, still in an algebraic language.

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Denef and van den Dries enriched the theory with analytic functions on $\mathbb{Z}_p^n$ in [12]. Cell decomposition in the Denef - van den Dries language was proven by C in [2] and made uniform by C - Robinson - Lipshitz [5] and in C - Lipshitz [4].

Description of references

Kontsevich invented motivic integration and presented it in a talk [17], but didn't write it up, as an improvement on a result by Batyrev [1].

Denef and Loeser developed the theory and expanded it [13], [14],...

Hales notes the power of motivic integration towards the Langlands program [15].

Loeser, Sebag, Nicaise developed it towards formal schemes [19].

Description of references

C - Loeser brought in parameter dependence, the additive character ψ and the generalized transfer principle [8], [7].

Hrushovski - Kazhdan captured the most possible information by establishing isomorphisms of Grothendieck rings [16], and incorporate most of C - Loeser.

The fundamental lemma is shown to fall under the scope of the transfer principle C - Loeser - Hales [3].

b-minimality might turn out useful [6]...

There are many related results by Veys and many other people.

Many good overviews have been written by Denef, Loeser, Hales, Veys, Gordon, ...
And many more...

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