A Course on Motivic Integration IV

Raf Cluckers

École normale supérieure, Paris
KULeuven, Belgium

MODNET Training Workshop
Model theory and Applications
La Roche, Belgium
20 - 25 April 2008

Talk 4

1. b-minimality and cell decomposition
2. Open questions
3. References

We propose here a possible way how to address most issues of our previous talks, but which is not fully explored yet.

Again we try to transmit as much intuition and goals as possible.

- *b*-minimality

What lacked for a long time in the theory of Henselian valued fields, was an axiomatic framework that implies cell decomposition (which says, roughly, that sets in one variable are nice combinations of balls and a measure zero set, also in families), suitable for integration.
In the notion of *b*-minimality, the first and most essential axiom, named (b1), guarantees cell decomposition in a rude way: in fact, axiom (b1) imposes almost literally cell decomposition on the theory.
b-minimality has left the realm of valued fields and became an abstract theory (which is in particular still useful for valued fields), consisting of three axioms (b1), (b2), (b3), where (b1) remains the essential one in most situations, and where (b2) and (b3) are included to have a good dimension theory in full generality.

We will mostly focus on (b1) and sketch when it implies (b2) and (b3).

Preliminaries:

We work with models $\mathcal{M}$ with many sorts and with one main sort, usually denoted by M. All other sorts are called auxiliary.

We suppose that there is a natural collection of balls, which are subsets of the main sort M.

Technically this collection is given by the fibers in M of a predicate $B \subset M \times A_B$ with A_B some definable set, depending on the theory one is looking at. Hence, the nonempty $B(a) := \{m \in M \mid (m, a) \in B\}$ form the natural collection of balls when a varies.

Definition

A model $\mathcal{M}$ is a (b1)-model if and only if
for each $\emptyset$-definable sets Y and $X \subset M \times Y$,
there exists a $\emptyset$-definable function $f : X \to Y \times S$,
with S purely living in the auxiliary sorts, that is, S is contained in a Cartesian product of the universes of auxiliary sorts,
such that for each (y, s) in $f(X)$, the fiber $f^{-1}(y, s) = E \times \{y\}$,
with E either a point (that is, a singleton) or a ball (that of course may depend on y and s).

Definition

A theory is a (b1)-theory if and only if all its models are.

If the main sort is a local field and the auxiliary sorts are countable, then (b1) implies (b2) and (b3).

More generally and more vaguely, if the main sort as well as every ball has uncountable cardinality, if the auxiliary sorts are countable, and if "'there are not too many different balls", then (b1) implies again (b2) and (b3).

Example:

- main sort $k((t))$ with k countable and auxiliary sorts countable.
- main sort $\mathbb{R}$ and auxiliary sorts countable.

However, (b1) is often very hard to prove!

Examples of (b1)-theories:

- Theory of Henselian valued fields of characteristic 0, using the Basarab - auxiliary sorts RV_n for each $n > 0$ which are $RV_n(K) := K^\times/1 + nM_K$ with M_K the maximal ideal. (see Denef, Pas, C - Loeser) (goes back to Cohen),
- the same structures enriched with a wide class of possible collections of analytic functions on the valuation rings (see C, C - Lipshitz - Robinson, C - Lipshitz)
- The field $\mathbb{R}$ with a predicate for the subset $2^{\mathbb{Z}}$, and with auxiliary sort the (countable) set $2^{\mathbb{Z}}$. (see van den Dries)
- ??? The function $\mathbb{R}^\times \to \mathbb{R} : x \mapsto \sin(\log|x|)$ on the real field yields a d-minimal structure (C. Miller). Is it b-minimal, that is, is it (b1)? It seems so but needs proof.

The idea of (b1) for integrational purposes is that the fibers which are points form a measure zero set, and the other (nonempty) fibers are balls that are easily measured,
hence yielding a sum over the auxiliary set S instead of an integral over M (thus in one variable), where Y plays the role of parameter domain.
By the example "Henselian valued fields of characteristic 0" this allows for all p-adic fields to be models hence removes the issues $(^\circ)$ and $(^{\circ\circ})$, that is, no ambiguity of working with $k((t))$ with k of char 0 when one is actually interested in p-adic fields.

The characters χ_a are not yet fully understood, certainly not when they co-appear with the character ψ in the same integral.

On the other hand, ψ is easily integrated on balls if it has linear argument.
By change of variables we can make the argument of ψ linear, as soon as definable functions are piecewise C^1.

If one has (b1) and piecewise C^1 (and if there are "centers", namely definable functions approximating the balls), then we can interpolate uniformly local integrals involving ψ.
Moreover, change of variables also holds under these conditions.

However, it is not clear how to define an interesting, purely geometric object that interpolates these integrals in the generality of *b*-minimality, other than the "geometric" object that merely retains finite counting (which one can conjecture implies having many geometric data as well...).

Cells

X as in axiom (b1) is called a (1)-cell (with parameters from Y, or "over" Y) with presentation f, when all fibers of f are balls, that is, of the form $E \times \{y\}$ with E a ball. Likewise, it is a (0)-cell when all fibers are singletons.

One builds cells further up by induction on the number of variables.

Lemma

The type of a cell is well defined.

Dimension

The dimension of $X \subset M^n \times Y$ is, as in the o-minimal definition, the maximal sum $\sum_{j=1}^{n} i_j$,
where the maximum is taken over all cells C included in X, and $(i_1, \ldots, i_n)$ is the type of the cell C.

Lemma

The dimension of X is well defined and satisfies all desirable properties.

Open questions

- Uniform theory of log_p

  $$\log_p(1+s) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{s^n}{n}$$

  ,
  converging on $1 + \mathcal{M}$,
  small p. Stil better: the Iwasawa logarithm.
- the χ_a and ψ at the same time.
- Issue $(^{\circ\circ\circ\circ})$: is the localisation used by C - Loeser canonical, and if so, is there an isomorphism in the C - Loeser framework resembling the isomorphism of Hrushovski - Kashdan?

Description of references

The possibility to apply model theoretical tools to study p-adic integrals was opened up by Macintyre's QE [18] and Cohen's [9] work on cell decomposition. This did the job for definable p-adic integrals for Denef [10] and in [11] he reproves Macintyre's QE and cell decomposition à la Cohen. Denef - Pas and Macintyre treat uniformity issues for cell decomposition and p-adic integrals, still in an algebraic language.

Denef and van den Dries enriched the theory with analytic functions on $\mathbb{Z}_p^n$ in [12]. Cell decomposition in the Denef - van den Dries language was proven by C in [2] and made uniform by C - Robinson - Lipshitz [5] and in C - Lipshitz [4].

Description of references

Kontsevich invented motivic integration and presented it in a talk [17], but didn't write it up, as an improvement on a result by Batyrev [1].

Denef and Loeser developed the theory and expanded it [13], [14],...

Hales notes the power of motivic integration towards the Langlands program [15].

Loeser, Sebag, Nicaise developed it towards formal schemes [19].

Description of references

C - Loeser brought in parameter dependence, the additive character ψ and the generalized transfer principle [8], [7].

Hrushovski - Kazhdan captured the most possible information by establishing isomorphisms of Grothendieck rings [16], and incorporate most of C - Loeser.

The fundamental lemma is shown to fall under the scope of the transfer principle C - Loeser - Hales [3].

b-minimality might turn out useful [6]...

There are many related results by Veys and many other people.

Many good overviews have been written by Denef, Loeser, Hales, Veys, Gordon, ...
And many more...

Victor V. Batyrev, *Stringy Hodge numbers of varieties with Gorenstein canonical singularities*, Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), World Sci. Publ., River Edge, NJ, 1998, pp. 1–32. MR MR1672108 (2001a:14039)

R. Cluckers, *Analytic p-adic cell decomposition and integrals*, Trans. Amer. Math. Soc. **356** (2004), no. 4, 1489 – 1499, arXiv:math.NT/0206161.

R. Cluckers, T. Hales, and F. Loeser, *Transfer principle for the fundamental lemma*, arXiv:0712.0708, To appear in a book edited by M. Harris on "Stabilisation de la formule des traces, variétés de Shimura, et applications arithmétiques", available at http://www.institut.math.jussieu.fr/projets/fa/bp0.html.

R. Cluckers and L. Lipshitz, *Fields with analytic structure*, (submitted), see http://www.dma.ens.fr/∼cluckers/.

R. Cluckers, L. Lipshitz, and Z. Robinson, *Analytic cell decomposition and analytic motivic integration*, Ann. Sci. École Norm. Sup. **39** (2006), no. 4, 535–568, arxiv:math.AG/0503722.

R. Cluckers and F. Loeser, *b-minimality*, to appear in Journal of Mathematical Logic, arXiv:math.LO/0610183.

______, *Constructible exponential functions, motivic Fourier transform and transfer principle*, to appear in Annals of Mathematics, arXiv:math.AG/0512022.

______, *Constructible motivic functions and motivic integration*, to appear in Inventiones Mathematicae, arxiv:math.AG/0410203.

P. J. Cohen, *Decision procedures for real and p-adic fields*, Comm. Pure Appl. Math. **22** (1969), 131–151.

J. Denef, *The rationality of the Poincaré series associated to the p-adic points on a variety*, Inventiones Mathematicae **77** (1984), 1–23.

______, *p-adic semialgebraic sets and cell decomposition*, Journal für die reine und angewandte Mathematik **369** (1986), 154–166.

J. Denef and L. van den Dries, *p-adic and real subanalytic sets*, Annals of Mathematics **128** (1988), no. 1, 79–138.

J. Denef and F. Loeser, *Germs of arcs on singular algebraic varieties and motivic integration*, Inventiones Mathematicae **135** (1999), 201–232.

______, *Definable sets, motives and p-adic integrals*, Journal of the American Mathematical Society **14** (2001), no. 2, 429–469.

T. Hales, *Orbital integrals are motivic*, Proc. Amer. Math. Soc. **133** (2005), no. 5, 1515–1525, arXiv:math.RT/0212236.

E. Hrushovski and D. Kazhdan, *Integration in valued fields*, Algebraic geometry and number theory, Progr. Math., vol. 253, Birkhäuser Boston, Boston, MA, 2006, pp. 261–405.

M. Kontsevich, *Lecture on motivic integration at Orsay*, (December 7, 1995).

A. Macintyre, *On definable subsets of p-adic fields*, Journal of Symbolic Logic **41** (1976), 605–610.

J. Nicaise and J. Sebag, *Invariant de serre et fibre de milnor analytique. (french)*, C. R. Math. Acad. Sci. **341** (2005), no. 1, 21–24.