The theory of exponential differential equations for constant groups

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Abstract

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Ax's theorem and its extension to constant semabelian varieties, with proof. The existential closedness theorem, with proof. Application to the first order theory of reducts of differential fields, without proof.

Outline

1. The Generalized Schanuel property
2. Differential forms
3. Reducts of differentially closed fields
4. Proof of Schanuel property
5. Existential closedness
6. Proof of Existential closedness

Outline

Commutative algebraic groups

In characteristic zero, every commutative connected algebraic group G factors uniquely as

$$0 \to \mathbb{G}_a{}^l \times \mathbb{G}_m{}^k \to G \to A \to 0$$

for some $l, k \in \mathbb{N}$, and some abelian variety A.

Classification

- Vector group $\iff A = 0, k = 0 \iff G \cong \mathbb{G}_a{}^l$
- Algebraic torus $\iff A = 0, l = 0 \iff G \cong \mathbb{G}_m{}^k$
- Abelian variety $\iff k = l = 0 \iff G$ is a complete variety
- Semiabelian variety $\iff l = 0 \iff$ no vector subgroups
- Nvq-group $\iff$ no vector quotients

Every semiabelian variety is an nvq-group.

Tangent bundles

Definition

The tangent space at the identity of an algebraic group G is called the Lie algebra of G, written LG. The tangent bundle of G is written TG.

Facts

- $LG \cong \mathbb{G}_a{}^n$, where $n = \dim G$
- When G is connected and commutative, $TG \cong LG \times G$, a canonical isomorphism of algebraic groups.
- If $H \subseteq G$ then $LH \subseteq LG$, and $TH = LH \times H \subseteq TG$
- If $G = \mathbb{G}_m{}^n$, $H \subseteq G$ given by $\prod_{i=1}^n y_i^{m_i} = 1$, then LH is given by $\sum_{i=1}^n m_i x_i = 0$
- L and T are functors on the category of algebraic groups.

Ax's theorem – version 1

Fix $\langle K; +, \cdot, \partial, C\rangle$ a differential field, of characteristic 0.

Theorem (Ax)

Suppose $n \geqslant 1$, $\partial x_i = \frac{\partial y_i}{y_i}$ for $i = 1, \ldots, n$, and $\mathrm{td}(x_1, y_1, \ldots, x_n, y_n / C) < n + 1$. Then there are $m_i \in \mathbb{Z}$, not all zero, such that $\prod_{i=1}^{n} y_i^{m_i} \in C$.

Geometric terms

- $\Gamma := \left\{ (x, y) \;\middle|\; \partial x = \frac{\partial y}{y} \right\}$
- $\mathrm{td}(\bar{x}, \bar{y}/C) < n + 1$ means $(\bar{x}, \bar{y})$ lies in V, some algebraic subvariety of $\mathbb{G}_a{}^n \times \mathbb{G}_m{}^n$, defined over C, with $\dim V < n + 1$.
- $\prod_{i=1}^{n} y_i^{m_i} \in C$ means $\bar{y}$ lies in a C-coset of a proper algebraic subgroup H of $\mathbb{G}_m{}^n$.
- Also $\sum_{i=1}^{n} m_i x_i \in C$, so $\bar{x}$ lies in a C-coset of LH.

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Generalized Schanuel Property

Theorem (Ax's theorem – version 2)

If $(x, y) \in \Gamma^n \cap V$, *$V$ defined $/C$ and* $\dim V < n + 1$, *then* $(x, y) \in TH(K) + \gamma$, *with H a proper algebraic subgroup of* $\mathbb{G}_m{}^n$ *and* $\gamma \in T\mathbb{G}_m{}^n(C)$.

- Let G be an nvq-group, defined $/C$, dimension n.
- $TG = LG \times G$, tangent bundle of G.
- $\Gamma_G \subseteq TG(K)$, the solution set of the exponential differential equation of G.

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Exponential differential equations

- For each commutative algebraic group G defined over C, there is a logarithmic derivative map

$$l\partial_G : G(K) \to LG(K)$$

- If $G \cong \mathbb{G}_a{}^n$, then $LG = G$, and $l\partial_G(x_1, \ldots, x_n) = (\partial x_1, \ldots, \partial x_n)$.
- If $\dim G = n$ then $LG \cong \mathbb{G}_a{}^n$.
- The exponential differential equation of G is

$$l\partial_{LG}(x) = l\partial_G(y)$$

- Γ_G is the solution set.

Outline

Differential forms

Let A be a C-algebra. The A-module $\Omega(A/C)$ of (Kähler) differentials has:

- Generators $\{da \mid a \in A\}$ subject to relations:
- $d(a+b) = da + db$,
- $d(ab) = a\,db + b\,da$, and
- $dc = 0$ for each $c \in C$, $a, b \in A$.

$A \xrightarrow{d} \Omega(A/C)$ is the universal derivation.

Example

$A = C[x]$, the coordinate ring of $\mathbb{G}_a$.
For $p(x) \in A$, $d(p(x)) = p'(x)dx$
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Differential maps

- If V is an affine variety over C with coordinate ring A, a K-point of V corresponds to a C-algebra homomorphism $A \xrightarrow{x} K$.
- $\Omega(A/C)$ is functorial in A, so x defines a map:

$$\begin{array}{ccc} \Omega(A/C) & \xrightarrow{x_*} & \Omega(K/C) \\ \omega & \longmapsto & \omega(x) \end{array}$$

- If V is a non-affine variety (e.g. abelian variety), we still have the module $\Omega[V/C]$ of global differentials, and for each point x of $V(K)$, the map:

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- Fixing $\omega \in \Omega[V/C]$ and letting x vary, we have:

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The logarithmic derivative

- If $A = K$, a field, then $\Omega(K/C)$ is a K-vector space. Its dual space is $\mathrm{Der}(K/C)$.
- For $\partial \in \mathrm{Der}(K/C)$ and $\omega(x) \in \Omega(K/C)$, we have $\partial^* \omega(x) \in K$.
- For a commutative algebraic group G, $\Omega[G/C]$ is spanned over the ring $C[G]$ by a basis of invariant differential forms, $\zeta_1, \ldots, \zeta_n$.
- $l\partial_G(x) = \langle \partial^* \zeta_1(x), \ldots, \partial^* \zeta_n(x) \rangle$ (after choosing a basis for LG)
- Take (ζ_i), (ξ_i), corresponding bases of invariant forms on G, LG respectively. Let $\omega_i(x, y) = \zeta_i(y) - \xi_i(x)$. Then

$$\Gamma_G = \{(x, y) \mid \partial^* \omega_i(x, y) = 0 \text{ for } i = 1, \ldots, n\}$$

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Outline

Reducts of differentially closed fields

- Take $\langle K; +, \cdot, \partial, C\rangle \models \mathrm{DCF}_0$
- Let $\mathcal{S}$ be a countable collection of semiabelian varieties, each defined over $C_0 \subseteq C$, closed under products, connected subgroups, quotients, and under isogeny.
- Consider $K_{\mathcal{S}} = \langle K; +, \cdot, C, (\Gamma_G)_{G \in \mathcal{S}}, (\hat{c})_{c \in C_0}\rangle$

Axioms – 1: Algebraic Properties

For each $G, H \in \mathcal{S}$,

- U1 K is an algebraically closed field, C is a (relatively) algebraically closed subfield, and the subfield C_0 of C is named by parameters.
- U2 Γ_G is a subgroup of TG.
- U3 $TG(C) \subseteq \Gamma_G$
- U4 $(0, y) \in \Gamma_G \iff y \in G(C)$ and $(x, 1) \in \Gamma_G \iff x \in LG(C)$, where $0, 1$ are the identities of LG, G.
- U5 If $G \xrightarrow{f} H$ is an algebraic group homomorphism then $(Tf)(\Gamma_G) \subseteq \Gamma_H$, and if f is an isogeny then also $\Gamma_G = (Tf)^{-1}(\Gamma_H)$.
- U6 If $G \subseteq H$ then $\Gamma_G = \Gamma_H \cap TG$.
- U7 $\Gamma_{G \times H} = \Gamma_G \times \Gamma_H$.

Axioms – 2: Other axioms

SP If $(x, y) \in \Gamma_G \cap V$, V defined $/C$ and $\dim V < \dim G + 1$, then $(x, y) \in TH(K) + \gamma$, with H a proper algebraic subgroup of G and $\gamma \in TG(C)$.

EC For each $G \in \mathcal{S}$ and each rotund subvariety V of TG, the intersection $\Gamma_G \cap V$ is nonempty.

Non-Triviality $K \neq C$

Theorem

The axioms U1 – U7, SP, EC, and NT hold in $K_{\mathcal{S}}$, they are first-order expressible, and they give its complete first order theory.

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Step 1: Setup

Theorem (Schanuel Property)

If $(x, y) \in \Gamma_G$ *and* $\mathrm{td}(x, y/C) < \dim G + 1$ *then there is a proper algebraic subgroup* H *of* G *and* $\gamma \in TG(C)$ *such that* (x, y) *lies in the coset* $TH + \gamma$*.*

- Let $(x, y) \in \Gamma_G$ with $\mathrm{td}(x, y/C) < n + 1$, $n = \dim G$.
- We have $\partial^* \omega_i(x, y) = 0$ for $i = 1, \ldots, n$.

Step 2: K-linear dependence of $\omega_i(x, y)$

Let $E = C(x, y)$ be the field generated by x, y. Then

- $\dim(\Omega(E/C)) = \mathrm{td}(E/C) \leqslant n$
- $\mathrm{Ann}(\partial) \subseteq \Omega(E/C)$, a subspace of codimension 1
- $\omega_i(x, y) \in \Omega(E/C) \cap \mathrm{Ann}(\partial)$, each i
- So the $\omega_i(x, y)$ are E-linearly dependent.
- In particular, they are K-linearly dependent.

Let $\alpha_i \in K$, such that $\sum_{i=1}^{n} \alpha_i \omega_i(x, y) = 0$, with the least possible number of non-zero coefficients. Without loss of generality, $\alpha_1 = 1$.

Step 3: C-linear dependence of $\omega_i(x, y)$

Fact

There is a Lie Derivative $L_\partial = d\partial^* + \partial^* d$ which acts like a derivative on differential forms.

$$\begin{aligned} 0 &= L_\partial \sum_{i=1}^{n} \alpha_i \omega_i(x, y) = \sum_{i=1}^{n} (L_\partial \alpha_i)\omega_i(x, y) + \alpha_i L_\partial \omega_i(x, y) \\ &= \sum_{i=1}^{n} (\partial \alpha_i)\omega_i(x, y) + \alpha_i(d\partial^* \omega_i(x, y) + \partial^* d\omega_i(x, y)) \\ &= \sum_{i=1}^{n} (\partial \alpha_i)\omega_i(x, y) \end{aligned}$$

Now $\partial \alpha_1 = \partial 1 = 0$, so by minimality of the α_i, all $\partial \alpha_i = 0$. Hence each $\alpha_i \in C$.

Step 4: subgroup of TG

- Let $\eta = \sum_{i=1}^{n} \alpha_i \omega_i \in \Omega[TG]$. We have

$$\begin{array}{ccc} TG(K) & \xrightarrow{\eta} & \Omega(K/C) \\ (x,y) & \longmapsto & \eta(x,y) \end{array}$$

  which is a group homomorphism.
- The ω_i are linearly independent in $\Omega[TG]$, hence $\eta \neq 0$. So $\ker \eta$ is a proper subgroup of $TG(K)$.
- Let $V = \mathrm{Loc}_C(x,y)$, V' a translate containing the identity, (x',y') the translate of (x,y).
- By Chevalley/Zilber indecomposability, there is m such that $mV' = A$, an algebraic subgroup of TG.
- Let O be orbit of (x',y') under $\mathrm{Aut}(K/C)$.
- $mO \subseteq mV'K$, containing all the generics of the group A.
- $2mO = A$, as every group element is a product of two generics.
- $O \subseteq \ker \eta$, so A is a proper algebraic subgroup of TG.
- Hence (x,y) lies in a C-coset of A.

Step 5: subgroup of G

- We assume that G is a semiabelian variety. (Piotr will extend the proof to nvq-groups and further.)
- It follows that every algebraic subgroup of TG is of the form $J \times H$, with $J \subseteq LG$ and $H \subseteq G$.
- For any $h \in H$,

$$\sum_{i=1}^{n} \alpha_i \zeta_i(h) = \eta(0, h) = 0$$

  because $(0, h) \in A \subseteq \ker \eta$.
- Thus $H \subseteq \ker \sum_{i=1}^{n} \alpha_i \zeta_i$.
- The ζ_i are linearly independent, hence H is a proper algebraic subgroup of G.
- Hence y lies in a constant coset of a proper algebraic subgroup H of G.

Step 6: constant cosets

- The quotient group $\Gamma_G / TG(C)$ is the graph of a bijection

$$\frac{\mathrm{pr}_{LG}\Gamma_G}{LG(C)} \xrightarrow{\theta} \frac{\mathrm{pr}_G \Gamma_G}{G(C)}$$

- $$\theta^{-1}((\mathrm{pr}_G \Gamma_G \cap H) \cdot G(C)) = (\mathrm{pr}_{LG} \Gamma_G \cap LH) \cdot LG(C)$$

- Hence x lies in a constant coset of LH. Thus (x, y) lies in a constant coset of TH, as required.

That completes the proof of the Schanuel property.

Outline

Strong Rotundity

Schanuel Property

If $(x, y) \in \Gamma_G \cap V$, V defined $/C$, and (x, y) does not lie in a C-coset of $TH(K)$ for a proper algebraic subgroup H of G, then $\dim V \geqslant n + 1$.

But also...

If $G \xrightarrow{f} J$ is a quotient map, then $Tf(x, y) \in \Gamma_J \cap (Tf)(V)$, so $\dim(Tf)(V) \geqslant \dim J + 1$ (or $J = 0$).

Definition

$V \subseteq TG$, irreducible and defined $/C$, is **strongly rotund** iff for each non-zero quotient $G \xrightarrow{f} J$ we have $\dim(Tf)(V) \geqslant \dim J + 1$.

For V strongly rotund, SP does not place a bar on $\Gamma_G \cap V$ having non-constant solutions.

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- $V \subseteq TG$, irreducible, not necessarily defined $/C$, is rotund iff for each quotient $G \xrightarrow{f} J$ we have $\dim(Tf)(V) \geqslant \dim J$.

Theorem (Existential closedness)

For each $G \in \mathcal{S}$ and each rotund subvariety V of TG, the intersection $\Gamma_G \cap V$ is nonempty.

Corollary

If V is strongly rotund and defined over C then $\Gamma_G \cap V$ has a non-constant point.

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Step 1: Extending derivations

$\langle K; +, \cdot, \partial, C\rangle \models \mathrm{DCF}_0$. Let $V \subseteq TG$ be rotund, defined over K. We show $\Gamma_G \cap V(K) \neq \emptyset$.

- By intersecting with generic hyperplanes, we can assume $\dim V = n := \dim G$.
- Let (x, y) be generic in V over K, and consider the field $K' = K(x, y)$.
- Define $\mathrm{Der}(K'/\partial) := \{\delta \in \mathrm{Der}(K'/C) \mid \exists \lambda \in K', \delta\restriction_K = \lambda\partial\}$

We have a sequence of inclusions of K'-vector spaces

$$\mathrm{Der}(K'/K) \hookrightarrow \mathrm{Der}(K'/\partial) \hookrightarrow \mathrm{Der}(K'/C)$$

and dually surjections

$$\Omega(K'/C) \twoheadrightarrow \Omega(K'/\partial) \twoheadrightarrow \Omega(K'/K)$$

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- By intersecting with generic hyperplanes, we can assume $\dim V = n := \dim G$.
- Let (x,y) be generic in V over K, and consider the field $K' = K(x,y)$.
- Define $\mathrm{Der}(K'/\partial) := \{\delta \in \mathrm{Der}(K'/C) \mid \exists \lambda \in K', \delta\restriction_K = \lambda\partial\}$

We have a sequence of inclusions of K'-vector spaces

$$\mathrm{Der}(K'/K) \hookrightarrow \mathrm{Der}(K'/\partial) \hookrightarrow \mathrm{Der}(K'/C)$$

and dually surjections

$$\Omega(K'/C) \twoheadrightarrow \Omega(K'/\partial) \twoheadrightarrow \Omega(K'/K)$$

Step 2: Linear algebra

- The differential forms $\omega_i(x, y)$ defining Γ_G lie in $\Omega(K'/C)$.
- Since V is rotund and (x, y) is generic in V over K, steps 3–6 of the proof of SP show that the images of the $\omega_i(x, y)$ in $\Omega(K'/K)$ are K'-linearly independent.
- Thus the images in $\Omega(K'/\partial)$ and $\Omega(K'/C)$ are K'-linearly independent, that is, their span Λ has dimension n.
- Thus the annihilator $\mathrm{Ann}(\Lambda)$ has codimension n in $\mathrm{Der}(K'/\partial)$ and in $\mathrm{Der}(K'/K)$.
- $\dim \Omega(K'/\partial) = \dim \mathrm{Der}(K'/\partial) = n + 1$, $\dim \mathrm{Der}(K'/K) = n$.
- Hence $\dim \mathrm{Der}(K'/\partial) \cap \mathrm{Ann}(\Lambda) = 1$, $\dim \mathrm{Der}(K'/K) \cap \mathrm{Ann}(\Lambda) = 0$.
- So there is $\delta \in \mathrm{Der}(K'/\partial) \cap \mathrm{Ann}(\Lambda) \setminus \mathrm{Der}(K'/K)$.

Step 3: K is differentially closed

- We have $\delta \in \mathrm{Der}(K'/\partial) \cap \mathrm{Ann}(\Lambda) \setminus \mathrm{Der}(K'/K)$.
- $\delta\restriction_K = \lambda\partial$ for some non-zero λ, so $\lambda^{-1}\delta$ extends ∂.
- In the differential field $\langle K', +, \cdot, \lambda^{-1}\delta, C\rangle$, we have $(x, y) \in \Gamma_G \cap V$.
- K is differentially closed, so $\Gamma_G \cap V$ is non-empty in K.

That completes the proof of existential closedness.