On the Ax-Schanuel property in arbitrary characteristic

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Set-up

In this characteristic 0 section we have:

- (K, ∂) a differential field of characteristic 0.
- $C := \partial^{-1}(0)$ the field of constants.
- A, B commutative algebraic groups defined over C.
- $G := A \times B$.
- $\dim(A) = \dim(B) := n$.
- LA is the tangent space of A at 0. L is a functor – for any algebraic homomorphism $f : A \to B$, we have $Lf : LA \to LB$.
- $l\partial_A : A(K) \to LA(K)$ the logarithmic derivative map.
- $\ker(l\partial_A) = A(C)$.

The differential equation of an algebraic map

Assume $f : A \to B$ is an algebraic homomorphism defined over C.

The following diagram is commutative (naturality of $I\partial$):

$$\begin{array}{ccc} LA(K) & \xrightarrow{Lf} & LB(K) \\ {\scriptstyle I\partial_A}\uparrow & & \uparrow{\scriptstyle I\partial_B} \\ A(K) & \xrightarrow{f} & B(K) \end{array}$$

The differential equation of f

- The differential equation of f is $Lf(I\partial_A(x)) = I\partial_B(y)$.
- Its solution set Γ_f is a (K, ∂)-definable subgroup of $G(K)$.
- By the diagram above and $\ker(I\partial_G) = G(C)$, we have $\Gamma_f = \mathrm{graph}(f) + G(C)$: the graph of f blurred by constants.

Local analytic maps

Assume $\mathbb{C} \subseteq C$ and A, B are defined over $\mathbb{C}$.

Complex Lie groups

- $A(\mathbb{C}), B(\mathbb{C})$ are complex manifolds with group operations given by holomorphic maps, so they are complex Lie groups.
- Let $f : A(\mathbb{C}) \to B(\mathbb{C})$ be a (local) analytic homomorphism. Then we still have a $\mathbb{C}$-linear map $Lf = f_0' : LA(\mathbb{C}) \to LB(\mathbb{C})$.
- By tensoring, we get a K-linear map $Lf : LA(K) \to LB(K)$.

Example

$\exp : \mathbb{G}_{\mathrm{a}}(\mathbb{C}) \to \mathbb{G}_{\mathrm{m}}(\mathbb{C})$ is an analytic homomorphism. Since $\mathbb{G}_{\mathrm{a}}(\mathbb{C}), \mathbb{G}_{\mathrm{m}}(\mathbb{C})$ are open in $\mathbb{C}$, we identify $L\mathbb{G}_{\mathrm{a}}(\mathbb{C}) = \mathbb{C} = L\mathbb{G}_{\mathrm{m}}(\mathbb{C})$. Since $\exp'(0) = \exp(0) = 1$, we have $L\exp = \mathrm{id}$.

The differential equation of an analytic map

Remark

There is no map $f : A(K) \nrightarrow B(K)$, since there is no convergence notion in K! Therefore, we only have the following diagram:

$$\begin{array}{ccc} LA(K) & \xrightarrow{Lf} & LB(K) \\ {\scriptstyle l\partial_A}\uparrow & & \uparrow{\scriptstyle l\partial_B} \\ A(K) & & B(K) \end{array}$$

The differential equation of f

- The differential equation of f is $Lf(l\partial_A(x)) = l\partial_B(y)$.
- Its solution set Γ_f is a (K, ∂)-definable subgroup of $G(K)$.
- Γ_f may be thought of as the graph of the non-existing $f : A(K) \nrightarrow B(K)$ blurred by constants.

Some analytic maps and their differential equations

Example (Analytic maps)

1. Any algebraic map $f : A(\mathbb{C}) \to B(\mathbb{C})$.
2. There exists an analytic epimorphism $\exp_B : LB(\mathbb{C}) \to B(\mathbb{C})$. Topologically, $\exp_B$ is the universal covering map.
3. For $\gamma \in \mathbb{C}$, the local map $\mathbb{G}_{\mathrm{m}}(\mathbb{C}) \ni x \mapsto x^\gamma \in \mathbb{G}_{\mathrm{m}}(\mathbb{C})$.
4. For E an elliptic curve, $\exp_E : LE(\mathbb{C}) \to E(\mathbb{C})$ factors through $\mathbb{G}_{\mathrm{m}}(\mathbb{C})$ to give an analytic epimorphism $r : \mathbb{G}_{\mathrm{m}}(\mathbb{C}) \to E(\mathbb{C})$.

Example (Differential equations)

1. $\Gamma_f = \mathrm{graph}(f) + G(C)$.
2. $\Gamma_{\exp_B} = \Gamma_B$ (Γ_B as in Jonathan's talk).
3. $\Gamma_{x \mapsto x^\gamma} : \frac{\partial x}{x} = \gamma \frac{\partial y}{y}$.
4. $\Gamma_r : \frac{\partial x}{x} = \frac{\partial y}{z}$, where $(y, z) \in E(K)$.

The differential equation of the exponential map

Example (An analytic solution)

Let $\mathcal{A}(\mathbb{C})$ be the ring of holomorphic functions on $\mathbb{C}$ with a natural derivation ∂_z and $\mathcal{M}(\mathbb{C})$ its field of fractions (meromorphic functions). Then $\exp \in \mathcal{M}(\mathbb{C})$ and $\partial_z(\exp) = \exp$, so the pair $(\mathrm{id}_{\mathbb{C}}, \exp)$ satisfies the differential equation of the exponential map.

Example (A formal solution)

We consider $C[\![X]\!]$, $\exp(X) := \sum \frac{X^i}{i!}$ and a natural derivation

$$\partial(\sum_{i=0}^{\infty} a_i X^i) = \sum_{i=0}^{\infty} a_{i+1}(i+1)X^i.$$

$(X, \exp(X))$ satisfies the differential equation of the exponential map.

Definition of the Ax-Schanuel property

Let $f : A(\mathbb{C}) \to B(\mathbb{C})$ be a local analytic homomorphism.

Definition

We say that Γ_f has the Ax-Schanuel property if

- $(a, b) \in \Gamma_f$ and $\mathrm{trdeg}_C(a, b) \leqslant n$

implies

- there are proper algebraic subgroups $A_0 < A, B_0 < B$ over C such that $a \in A_0(K) + A(C)$ and $b \in B_0(K) + B(C)$.

Remark

- If $A = \mathbb{G}_{\mathrm{a}}^n$ and there is $B_0 \lneqq B$ such that $b \in B_0(K) + B(C)$, then there is $A_0 \lneqq A$ such that $a \in A_0(K) + A(C)$. Hence having B_0 alone is enough to get the Ax-Schanuel property.
- If $f = \exp_B$, then we can take $A_0 = LB_0$ as in Jonathan's talk.

Formal maps and the Ax-Schanuel property

- $\exp_{\mathbb{G}_{\mathrm{m}}^n}$ has the Ax-Schanuel property (Ax's theorem).
- $\exp_A$ has the Ax-Schanuel property for A semi-abelian (Jonathan's talk) and even for A with no vectorial quotients.
- Any analytic epimorphism from an algebraic torus to an abelian variety has the Ax-Schanuel property.
- Raising to power γ on $\mathbb{G}_{\mathrm{m}}^n$ has the Ax-Schanuel property, if $[\mathbb{Q}(\gamma) : \mathbb{Q}] > n$.
- Raising to power $\sqrt{2}$ on $\mathbb{G}_{\mathrm{m}}^2$ DOES NOT have the Ax-Schanuel property.
- Algebraic maps $f : A \to B$ DO NOT have the Ax-Schanuel property. Take $a \in A(K)$ which is NOT in a constant coset of a proper subgroup. Still $(a, f(a)) \in \Gamma_f$, $\operatorname{trdeg}_C(a, f(a)) \leqslant n$.

About proofs of the Ax-Schanuel property

The proofs (except the raising to power case) do not differ much from the one given in Jonathan's talk.

- One gets a C-algebraic subgroup $H < A \times B$ such that WLOG $(a, b) \in H(K), \quad H_A := H \cap A \neq A, \quad H_B := H \cap B \neq B.$
- In many cases H has to be of the form $H_A \times H_B$, done.
- Let us deal with the case $\exp_B$, B has no vectorial quotients.
- Let H^B be the image of H by projection to B (similarly H^A). It is enough to show $H^B \neq B$. Assume $H^B = B$.
- Then H induces an algebraic epimorphism $H^A \twoheadrightarrow B/H_B$.
- But $H^A \leqslant A = LB \cong \mathbb{G}_{\mathrm{a}}^n$, so H^A is a vector group.
- Hence B/H_B is a vector group, being an image of H^A and nontrivial, so B has a vectorial quotient, a contradiction.

Hasse-Schmidt derivations needed

Problem

Assume $\mathrm{char}(K) = p > 0$. Then for each $x \in K^p$, $\partial(x) = 0$, so K is algebraic over $C \supseteq K^p$ and the statement of Ax is meaningless, since $\mathrm{trdeg}_C(x)$ is always 0.

Solution

Replace the derivation ∂ with a Hasse-Schmidt derivation (HS-derivation) $D = (D_n : K \to K)_{n<\omega}$, i.e.

1. D_0 is the identity map,
2. each D_n is additive,
3. $D_n(xy) = \sum_{i+j=n} D_i(x)D_j(y)$ (Leibniz rule),
4. $D_i \circ D_j = \binom{i+j}{i} D_{i+j}$ (iterativity condition).

Let C be the constant field of all D_n, i.e. $C := \bigcap_{n>0} \ker(D_n)$. Then K is usually not algebraic over C.

What has really happened?

Derivations as homomorphisms

A map $\partial : K \to K$ is a derivation if and only if the map $K \ni a \mapsto a + \partial(a)X \in K[X]/(X^2)$ is a ring homomorphism.

HS-derivations as homomorphisms

A sequence of maps $D = (D_i : K \to K)_{i<\omega}$ is an HS-derivation if and only if $D_0 = \mathrm{id}_K$, the map $K \ni a \mapsto \sum D_i(a)X^i \in K[\![X]\!]$ is a ring homomorphism and D satisfies the iterativity condition.

Moral

We always have to make the following replacement:

$$K[X]/(X^2) \rightsquigarrow K[\![X]\!].$$

What happens to the tangent space after the replacement?

Let V be a C-variety and R a C-algebra.

A new interpretation of the tangent space

The tangent space TV satisfies the following canonical bijection

$$TV(R) \longleftrightarrow V(R[X]/(X^2)).$$

The space of arcs

- The n^{th}-arc space of V, $\mathrm{Arc}^n(V)$, is a variety satisfying the following canonical bijection

$$\mathrm{Arc}^n(V)(R) \longleftrightarrow V(R[X]/(X^{n+1})).$$

- Since $R[\![X]\!] = \varprojlim R[X]/(X^n)$, the role of the tangent space is played by the full arc space $\mathrm{Arc}(V) := \varprojlim \mathrm{Arc}^n(V)$, which is a pro-algebraic variety. Note that $\mathrm{Arc}^1(V) = TV$.

What happens to LG after the replacement?

Let G be a commutative algebraic group over C.

Relation between LG and TG

Recall from Jonathan's talk:

1. TG is a commutative algebraic group.
2. The projection map $TG \to G$ is a group homomorphism.
3. LG is the fiber over 0 of the projection map.
4. TG canonically decomposes as $LG \times G$.

U_G is the HS-replacement of LG

1. $\mathrm{Arc}(G)$ is a commutative pro-algebraic group.
2. The projection map $\mathrm{Arc}(G) \to G$ is a group homomorphism.
3. Let U_G denote the fiber over 0 of the map $\mathrm{Arc}(G) \to G$.
4. $\mathrm{Arc}(G)$ canonically decomposes as $U_G \times G$.

More on $\mathrm{Arc}(G)$ and U_G

Let G be a commutative algebraic group over C and $m < \omega$.

U_G is pro-unipotent

- Let U_G^m denote the fiber over 0 of the map $\mathrm{Arc}^m(G) \to G$.
- There is an exact sequence of algebraic groups:

$$0 \to LG \to U_G^{m+1} \to U_G^m \to 0.$$

- Therefore U_G^m is unipotent, so U_G is pro-unipotent.

Remark

One computes the group law of $\mathrm{Arc}^m(G)$ by formally HS-differentiating the group law of G.

The group laws of $U^m_{\mathbb{G}_a}$ and $U_{\mathbb{G}_a}$

- Start from $X + Y$, the group law of $\mathbb{G}_a$.
- Compute formally: $D_i(X + Y) = D_i(X) + D_i(Y),\ i \leqslant m$.
- Let us denote the variables related to $\mathrm{Arc}^m(\mathbb{G}_a)$ by:

$$X = D_0(X),\ X' := D_1(X),\ \ldots\ ,\ X^{(m)} := D_m(X).$$

- The group law of $\mathrm{Arc}^m(\mathbb{G}_a)$ is given by:

$$(X, \ldots, X^{(m)}) + (Y, \ldots, Y^{(m)}) = (X + Y, \ldots, X^{(m)} + Y^{(m)}),$$

  so $\mathrm{Arc}^m(\mathbb{G}_a) \cong \mathbb{G}_a^{m+1}$.
- The group law of $U^m_{\mathbb{G}_a}$ is given by the group law in $\mathrm{Arc}^m(\mathbb{G}_a)$ AFTER plugging $X = Y = 0$ ($U^m_{\mathbb{G}_a}$ is the fiber over 0):

$$(0, X', \ldots, X^{(m)}) + (0, Y', \ldots, Y^{(m)}) = (0, X' + Y', \ldots, X^{(m)} + Y^{(m)}),$$

  so $U^m_{\mathbb{G}_a} \cong \mathbb{G}_a^m$ and $U_{\mathbb{G}_a} \cong \mathbb{G}_a^\infty$.

The group law of $U^m_{\mathbb{G}_m}$

- Start from XY, the group law of $\mathbb{G}_m$.
- Compute formally: $D_i(XY) = \sum_{k+l=i} D_k(X)D_l(Y),\ i \leqslant m$.
- Let $X, X', \ldots, X^{(m)}$ be the $\mathrm{Arc}^m(\mathbb{G}_m)$-variables as before.
- The group law of $\mathrm{Arc}^m(\mathbb{G}_m)$ is given by:

$$(X, X' \ldots, X^{(m)}) + (Y, Y', \ldots, Y^{(m)}) = \\ = (XY, X'Y + Y'X, \ldots, \textstyle\sum_{k+l=m} X^{(k)}Y^{(l)}).$$

- The group law of $U^m_{\mathbb{G}_m}$ is given by the group law of $\mathrm{Arc}^m(\mathbb{G}_m)$ AFTER plugging $X = Y = 1$ ($U^m_{\mathbb{G}_m}$ is the fiber over 1):

$$(1, X', \ldots, X^{(m)}) + (1, Y', \ldots, Y^{(m)}) = \\ = (1, X' + Y', \ldots, X^{(m)} + Y^{(m)} + \textstyle\sum_{i=1}^{m-1} X^{(i)}Y^{(m-i)}).$$

The group laws of $U_{\mathbb{G}_a}$ and $U_{\mathbb{G}_m}$

Another interpretation of $U_{\mathbb{G}_a}$

- $\mathrm{Arc}(\mathbb{G}_a)(C) = \mathbb{G}_a(C[\![X]\!])$.
- $U_{\mathbb{G}_a}(C) = \ker[\mathbb{G}_a(C[\![X]\!]) \to \mathbb{G}_a(C)]$.
- $U_{\mathbb{G}_a}(C) = (XC[\![X]\!], +) \cong \mathbb{G}_a^\infty(C)$.

Another interpretation of $U_{\mathbb{G}_m}$

- $U_{\mathbb{G}_m}(C) = \ker[\mathbb{G}_m(C[\![X]\!]) \to \mathbb{G}_m(C)]$.
- $U_{\mathbb{G}_m}(C) = (1 + XC[\![X]\!], \cdot) \cong \mathbb{G}_a(W(C))$.
- $W(C)$ is the ring of infinite Witt vectors over C. For example $W(\mathbb{F}_p) = \mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ – the ring of p-adic integers.
- Since $\mathrm{char}(W(C)) = 0$, the group $U_{\mathbb{G}_m}(C) \cong \mathbb{G}_a(W(C))$ is torsion-free, so $U_{\mathbb{G}_a} \not\cong U_{\mathbb{G}_m}$, since $U_{\mathbb{G}_a} \cong \mathbb{G}_a^\infty$ is p-torsion.

Full logarithmic derivative

Derivation case

1. $\partial : K \to K[X]/(X^2)$ gives a (K, ∂)-definable homomorphism

$$\partial_G : G(K) \to G(K[X]/(X^2)) = TG(K)$$

2. After composing with the projection map $TG(K) \to LG(K)$ we get the logarithmic derivative $l\partial_G : G(K) \to LG(K)$.

HS-derivation case

1. The HS-derivation $D : K \to K[\![X]\!]$ induces a $*$-definable in (K, D) homomorphism

$$D_G : G(K) \to G(K[\![X]\!]) = \mathrm{Arc}(G)(K).$$

2. After composing with the projection $\mathrm{Arc}(G)(K) \to U_G(K)$ we get the full logarithmic derivative $lD_G : G(K) \to U_G(K)$.

Full logarithmic derivatives on $\mathbb{G}_{\mathrm{a}}$ and $\mathbb{G}_{\mathrm{m}}$

Example ($\mathbb{G}_{\mathrm{a}}$)

The full logarithmic derivative on $\mathbb{G}_{\mathrm{a}}$ is:

$$\mathbb{G}_{\mathrm{a}}(K) \ni a \mapsto (D_i(a))_{i<\omega} \in U_{\mathbb{G}_{\mathrm{a}}}(K) = \mathbb{G}_{\mathrm{a}}^{\infty}(K).$$

Example ($\mathbb{G}_{\mathrm{m}}$)

The full logarithmic derivative on $\mathbb{G}_{\mathrm{m}}$ is:

$$\mathbb{G}_{\mathrm{m}}(K) \ni a \mapsto \left(\frac{D_i(a)}{a}\right)_{i<\omega} \in U_{\mathbb{G}_{\mathrm{m}}}(K) = \mathbb{G}_{\mathrm{a}}(W(K)).$$

Set-up

In the HS-differential case we have:

- (K, D) a field with an HS-derivation.
- $C := \bigcap_{i>0} \ker(D_i)$ the absolute field of constants.
- A, B commutative algebraic groups defined over C.
- $G := A \times B$.
- $\dim(A) = \dim(B) := n$.
- U_A is again a functor – for any algebraic homomorphism $f : A \to B$, we have a pro-algebraic map $U_f : U_A \to U_B$.
- $lD_A : A(K) \to U_A(K)$ the full logarithmic derivative map.
- $\ker(lD_A) = A(C)$.

How to replace a local analytic map (derivation case)?

- The differential equation came from a map $LA(K) \to LB(K)$ induced by a local analytic map $A(\mathbb{C}) \to B(\mathbb{C})$.
- Over any field C, we can talk about formal homomorphisms between A and B. For example $\exp(X) = \sum \frac{X^i}{i!}$ is a formal homomorphism between $\mathbb{G}_{\mathrm{a}}$ and $\mathbb{G}_{\mathrm{m}}$, because:

$$\exp(X+Y) = \sum_{i=0}^{\infty} \frac{(X+Y)^i}{i!} = \sum_{i=0}^{\infty} \frac{X^i}{i!} \cdot \sum_{i=0}^{\infty} \frac{Y^i}{i!} = \exp(X) \cdot \exp(Y).$$

- Formal homomorphisms between A and B still induce actual linear maps $LA \to LB$ (and there is a 1-to-1 correspondence).
- Therefore, we can replace local analytic homomorphisms by formal homomorphisms.

How to replace a local analytic map (HS-derivation case)?

Similarities between characteristic 0 and positive characteristic

Formal homomorphisms between A and B still induce pro-algebraic homomorphisms $U_A \to U_B$, so we can use them.

Differences between characteristic 0 and positive characteristic

- $U_{\mathbb{G}_a} \not\cong U_{\mathbb{G}_m}$, so in particular there is no formal isomorphism between $\mathbb{G}_a$ and $\mathbb{G}_m$. This means that the exponential map DOES NOT exist in positive characteristic.
- There may be more maps between U_A and U_B then formal homomorphisms between A and B. E.g. the ring of formal endomorphisms of $\mathbb{G}_m$ is $\mathbb{Z}_p = W(\mathbb{F}_p)$ (Manin) and the ring of pro-algebraic endomorphisms of $U_{\mathbb{G}_m}$ over C is $W(C)$.

Formal maps

Example

- Any sequence $(c_i \in C)_{i<\omega}$ gives a formal endomorphism of $\mathbb{G}_{\mathrm{a}}$

$$\sum_{i=0}^{\infty} c_i X^{p^i}.$$

- For any p-adic number $\sum_{i=0}^{\infty} a_i p^i$ the sequence

$$((X+1)^{\sum_{i=0}^{n} a_i p^i} - 1)_{n<\omega}$$

converges to a non-algebraic formal endomorphism of $\mathbb{G}_{\mathrm{m}}$.

- There is a formal isomorphism between $\mathbb{G}_{\mathrm{m}}$ and an ordinary elliptic curve E. The analytic epimorphism $r : \mathbb{G}_{\mathrm{m}}(\mathbb{C}) \to E(\mathbb{C})$ „survives“ to positive characteristic (unlike exp).

The HS-differential equation of a formal map

Assume f is a formal map between A and B and let $U_f : U_A \to U_B$ be the induced pro-algebraic homomorphism. Again, there is no map $f : A(K) \nrightarrow B(K)$, so we only have the following diagram:

$$\begin{array}{ccc} U_A(K) & \xrightarrow{U_f} & U_B(K) \\ {\scriptstyle ID_A}\uparrow & & \uparrow{\scriptstyle ID_B} \\ A(K) & & B(K) \end{array}$$

The HS-differential equation of f

- The HS-differential equation of f is $U_f(ID_A(x)) = ID_B(y)$.
- Its solution set $\Gamma_f \leqslant G(K)$ is type-definable in (K, D).
- Again, Γ_f may be thought of as the graph of the non-existing $f : A(K) \nrightarrow B(K)$ blurred by constants.

Some HS-differential equations

Example

We will hopefully compute on the board Γ_f for:

- $f = \sum_{i=0}^{\infty} c_i X^{p^i}$ on $\mathbb{G}_{\mathrm{a}}$.
- $f = \lim_n (X+1)^{\sum_{i=0}^{n} a_i p^i} - 1$ on $\mathbb{G}_{\mathrm{m}}$.

The HS-differential Ax-Schanuel property

Let f be a formal homomorphism between A and B.

Definition

We say that Γ_f has the Ax-Schanuel property if

- $(a, b) \in \Gamma_f$ and $\mathrm{trdeg}_C(a, b) \leqslant n$

implies

- there are proper algebraic subgroups $A_0 < A, B_0 < B$ over C such that $a \in A_0(K) + A(C)$ and $b \in B_0(K) + B(C)$.

Not much is known

Proving the HS-differential Ax-Schanuel property for certain formal homomorphisms is work in progress. The candidates are:

- $f = \sum_{i=0}^{\infty} c_i X^{p^i}$ on $\mathbb{G}_{\mathrm{a}}$.
- $f = \lim_n (X+1)^{\sum_{i=0}^{n} a_i p^i} - 1$ on $\mathbb{G}_{\mathrm{m}}$.
- A formal isomorphism f between $\mathbb{G}_{\mathrm{m}}$ and an ordinary elliptic curve E. This is the best candidate, since it is NOT a limit of algebraic homomorphisms between $\mathbb{G}_{\mathrm{m}}$ and E (each such is 0), so f is ,,very" non-algebraic.

Vojta's HS-differential forms

In the proof from Jonathan's talk the module of differential forms Ω was crucial. Here, the role of Ω is played by the graded ring of HS-differential forms introduced by Vojta.
Let $m < \omega$, V be an affine C-variety and R be a C-algebra.

HS-differential forms

- $\mathrm{HS}^m_{R/C}$ is the R-algebra generated by the set of symbols $\{\mathrm{d}_i a \mid i \leqslant m,\ a \in R\}$ and subject to the relations:
  1. $\mathrm{d}_i(x+y) = \mathrm{d}_i x + \mathrm{d}_i y, \quad x, y \in R, \quad i \leqslant m;$
  2. $\mathrm{d}_i(c) = 0, \quad c \in C;$
  3. $\mathrm{d}_i(xy) = \sum_{k+l=i} \mathrm{d}_k x \cdot \mathrm{d}_l y, \quad x, y \in R, \quad i \leqslant m.$
- $\mathrm{HS}^1_{R/C} \cong S(\Omega_{R/C})$ – the symmetric algebra.
- $C[\mathrm{Arc}^m(V)] \cong \mathrm{HS}^m_{C[V]/C}$.
- In particular $C[TV] \cong \mathrm{HS}^1_{C[V]/C} = S(\Omega_{C[V]/C})$.

HS-forms and the Ax-Schanuel property

Time permitting, I will try to explain the connection between the HS-forms and the Ax-Schanuel property.