# Model Theory of Valued Fields 

## §1 Basics of ACVF

Themes in model theory of valued fields

- Ax-Kochen/Ershov (1965)
- QE, decidability, axioms for various valued fields
- p-adic integration, rationality of zeta functions, motivic integration

- Algebraically closed valued fields ⟸
  (QE for ACVF, A. Robinson, 1956)
- model theory and rigid analytic geometry; $\mathcal{L}_{an}$

View ACVF as providing a universal domain for all valued fields.

We study quantifier-free definability in all valued fields.

## Basic Setting.

Field $K$, ordered abelian group $\Gamma$.

A valuation is a map $v : K \to \Gamma \cup \{\infty\}$

s.t.

(i) $v(x) = \infty \iff x = 0$

(ii) $v(xy) = v(x) + v(y)$ (so $v$ gives a hom. $(K^*, \cdot) \to (\Gamma, +)$ )

(iii) $v(x+y) \geqslant \mathrm{Min}\{v(x), v(y)\}$.

Valuation ring $R := \{x \in K : v(x) \geqslant 0\}$

has unique maximal ideal $\mathcal{M} := \{x \in K : v(x) > 0\}$

Residue field $k := R/\mathcal{M}$.

$(\mathrm{char}(K), \mathrm{char}(k)) \in \{(0,0), (0,p), (p,p)\}$.

Mostly assume $K$ is algebraically closed,

often assume $v$ is surjective.

But later we view $K, \Gamma$ as <u>sorts</u>.

**Lemma 1.3.1**

Let $(K, v, \Gamma)$ be an algebraically closed valued field ($v$ surjective). Then

(i) $\Gamma$ is divisible

(ii) $k$ is algebraically closed

**Pf:** (i) Let $\gamma \in \Gamma$, say $\gamma = v(x)$. Let $n \in \mathbb{N}^{>0}$. As $K$ is alg. closed, $\exists y \in K$ with $y^n = x$. Then $n v(y) = x$, so $v(y) = \frac{1}{n} \gamma$.

(ii) Ex.

**Examples 1.3.2** (i) Take any valued field $(L, v, \Gamma)$, and extend $v$ to algebraic closure $\tilde{L}$.

e.g. $\tilde{\mathbb{Q}}_p$ has a valuation to the divisible hull of $(\mathbb{Z}, <, +)$, namely $(\mathbb{Q}, <, +)$, with res. field $\tilde{\mathbb{F}}_p$. Can then extend the val. to its completion $\mathbb{C}_p$ (still alg. closed).

Example (ii) Hahn product – generalised power series.

Fix field $k$, ordered ab. gp $\Gamma$, indeterminate $t$.

Consider expansions

$$f = \sum_{\gamma \in \Gamma} a_\gamma t^\gamma \qquad (a_\gamma \in k)$$

Define $\operatorname{supp}(f) = \{ \gamma \in \Gamma : a_\gamma \neq 0 \}$.

$$K := k((t^\Gamma)) := \{ f = \sum_{\gamma \in \Gamma} a_\gamma t^\gamma : \text{supp}(f) \text{ is well-ordered} \}.$$

Addition, multiplication as for ordinary power series. Multiplication is well-defined because of 'well-order' condition.

$$\left( \sum a_\gamma t^\gamma \right) . \left( \sum b_\gamma t^\gamma \right) = \sum c_\gamma t^\gamma$$

where $c_{\gamma_0} = \underbrace{\sum_{\gamma + \delta = \gamma_0} a_\gamma b_\delta}_{\text{a finite sum.}}$

$K = k((t^\Gamma))$ is a field.

Define $v: K \to \Gamma$ by

$$v(f) = \mathrm{Min}\,(\mathrm{supp}(f)).$$

The residue field is $k$.

Clearly $k \hookrightarrow K$, so $\mathrm{char}(K) = \mathrm{char}(k)$

<u>Proposition</u> If $k$ is algebraically closed, and $\Gamma$ is divisible (and non-trivial), then $k((t^\Gamma))$ is <u>algebraically</u> closed.

Various notions of 'completion' for valued fields, which coincide if the value group is $\mathbb{Z}$.

<u>Def</u> (Krull, F.K. Schmidt)

A valued field is <u>maximally</u> <u>complete</u> if it has no proper immediate extensions (i.e. every valued field proper extension increases the value group or residue field).

A theory of maximally complete valued fields was developed by Kaplansky ~ 1940's

- every valued field has an immediate max. complete extension
- gen. power series fields $k((t^\Gamma))$ are max. complete
- max. complete fields are henselian.

_Example:_ Puiseux series $\bigcup_n k((t^{\mathbb{Z}/n!}))$

where $\mathbb{Z}/n!$ is the group of rationals $a/n!$ $(a \in \mathbb{Z})$.

If $k = \mathbb{C}$ (or any alg. closed field of char 0)

get an alg. closed valued field.

(not maximally complete!)

## Quantifier - elimination

Many possible languages for valued fields

e.g.

$L_v$: $(+, -, \cdot, 0, 1, \; v(x) \leq v(y))$

can then define val. ring $R$, so units $U(R)$,

so $\Gamma = K^*/U$.

$xU < yU \iff \exists z \in R \;\; (xzU = yU)$

$L_\Gamma$ : 2 sorts $\overset{K}{(+, -, \cdot, 0, 1)}$ , $\overset{\Gamma}{(<, +, -, 0)}$

val map $v: K \to \Gamma$

$L_{k,\Gamma}$ : 3 sorts

$\overset{K}{(+, -, \cdot, 0, 1)}$ , $\overset{k}{(+, -, \cdot, 0, 1)}$ $\overset{\Gamma}{(<, +, -, 0)}$

$v: K \to \Gamma$

$\mathrm{Res}: K^2 \to k$

$$\mathrm{Res}(x, y) = \begin{cases} \mathrm{res}(xy^{-1}) & \text{if } v(x) \geq v(y) \\ 0 & \text{o'wise} \end{cases}$$

<u>Theorem 1·4·1</u> (i) In any of $L_v$, $L_\Gamma$, $L_{k,\Gamma}$,

a complete theory is axiomatised by sentences which express

(a) $K$ is algebraically closed

(b) $v(K)$ is non-trivial

(c) the characteristics of $K$, $k$

(and that $\Gamma$ is an o.a.g., $v$ a valuation, $v$ surjective / val., etc.)

(ii) Alg. closed valued fields have QE in each of $L_v$, & $L_\Gamma$, $L_{k,\Gamma}$.

(iii) Any valued field having QE in $L_v$ is algebraically closed

(iv) QE is uniform in characteristics

<u>Def.</u> If $D \subset M^{eq}$ is $\emptyset$-definable, we say $D$ is <u>stably embedded</u> if, for any $t$, any definable subset of $D^t$ is definable, uniformly in parameters, over $D$.

<u>Prop</u> 2.0.5 In ACVF we have

(i) $\Gamma$ is stably embedded

(ii) $k$ is " " .

(iii) $k$ is a 'pure' alg. closed field: any $\emptyset$-def subset of $k^n$ is definable over $\emptyset$ just in $(k, +, \times)$

(iv) $\Gamma$ is a pure d.o.a.g. (possibly expanded by constants)

(v) $k$ is strongly minimal, $\Gamma$ is o-minimal.

(vi) algebraic closure (acl) in $K$ is field-theoretic algebraic closure, as in ACF.

<u>Def</u>: $(K, v, \Gamma)$ a valued field,
$a \in K$, $\gamma \in \Gamma \cup \{\infty\}$.

Open ball: $B_{>\gamma}(a) := \{x \in K : v(x-a) > \gamma\}$ $(\gamma \neq \infty)$

Closed ball: $B_{\geq \gamma}(a) := \{x \in K : v(x-a) \geq \gamma\}$

(so $\{a\} = B_{\geq \infty}(a)$).

<u>NB</u>: $B_{>\gamma}(0)$ is an $R$-submodule of $K$,
and its cosets are the $B_{>\gamma}(a)$.
Likewise for $\geq$.

Topology $\mathcal{T}$ on $K$.

$\mathcal{T}_{\geq}$ : $\{B_{\geq\gamma}(a) : a \in K, \gamma \in \Gamma\}$ - basis of open sets

$\mathcal{T}_{>}$ : $\{B_{>\gamma}(a) : \text{———}\}$ - basis of open sets

$\mathcal{T}_{\geq} = \mathcal{T}_{>} = \mathcal{T}$, and non-singleton balls are <u>clopen</u>.

$K$ is a topological field, with $\mathcal{T}$.

$\mathbb{Q}_p$, $\mathbb{F}_p((t))$ are locally compact. ACVF has no loc. comp. models.

Prop 2.0.7 Let $(K, v, \Gamma) \models ACVF$

Then any definable subset of $K$ is a finite boolean combination of balls.

In fact, any def. subset of $K$ is a finite union of 'Swiss cheeses'

Swiss cheese: ball (possibly $K$) $\setminus$ finite union of ~~proper~~ sub-balls

The Swiss cheeses can be chosen canonically (Holly).

Ex: Given any two balls $B_1, B_2$, we have one of:

$B_1 \subseteq B_2$, $B_2 \subseteq B_1$, or

$B_1 \cap B_2 = \emptyset$.

# §2 Notions of minimality in valued fields

Idea: find analogues for valued fields (and their expansions) of o-minimality for ordered fields.

cf. **Cluckers–Loeser: b-minimality.**

<u>Recall</u>: – $T$ is <u>strongly minimal</u> if for every $M \models T$, every definable subset of $M$ is finite or cofinite (so quantifier-free def. just from $=$).

– $T = Th(M, <, \ldots)$ is <u>o-minimal</u> if for every $N \models T$, every def. subset of $N$ is a finite union of open intervals and singletons (so quant.-free definable just using $<$).

(In o-minimal case, suffices to check this in $M$).

<u>Defn</u> Suppose $\mathcal{L} \subset \mathcal{L}^+$ are languages, $T^+$ an $\mathcal{L}^+$-theory. Then $T^+$ is <u>$\mathcal{L}$-minimal</u> if for every $M^+ \models T^+$, every definable subset of $M^+$ is definable by quantifier-free $\mathcal{L}$-formulas.

Recall, if $K \models ACVF$, every def. subset of $K$ is a b.c. of balls (and is a finite union of Swiss cheeses). So we want a notion of minimality with balls replacing intervals. Follow Hrushovski-Kazhdan rather than M-Steinhorn, Haskell-M.

Define as our base theory (analogous to total orderings in the o-min. case) a theory $T_{um}$ of ultrametric spaces.

Two sorts, $VF$ and $\Gamma_\infty$.

Assume $\Gamma_\infty$ has constant symbol $\infty$, and a binary reln. symbol $<$, and there is a function symbol

$$val : VF^2 \to \Gamma_\infty.$$

Write $\Gamma$ for $\Gamma_\infty \setminus \{\infty\}$.

Above language denoted $\mathcal{L}$.

# Axioms of $T_{um}$

(i) $\Gamma_\infty$ is a D.L.O. under $<$ with no least element, greatest element $\infty$.

(ii) $\mathrm{val}(x,y) = \infty \iff x = y$

(iii) For each $\alpha \in \Gamma_\infty$, the relation '$\mathrm{val}(x,y) \geq \alpha$' is an equivalence relation $E^C_\alpha$, whose classes are called 'closed $\alpha$-balls'.
For $\alpha \in \Gamma$, '$\mathrm{val}(x,y) > \alpha$' is an equiv. reln. $E^O_\alpha$, classes called 'open $\alpha$-balls'.

(iv) $\alpha < \beta \Rightarrow E^C_\beta$ refines $E^C_\alpha$, $E^O_\beta$ refines $E^O_\alpha$

(v) If $\alpha \in \Gamma$ each closed $\alpha$-ball is a union of infinitely many distinct open $\alpha$-balls.

**Defn.** A complete theory $T^+$ over $L^+ \supseteq L$ is called <u>C-minimal</u> if for every $M^+ \models T^+$, every definable subset of the VF-sort of $M^+$ is a finite b.c. of balls.

NB. $T_{um}$ is a theory of $\Gamma$-valued ultrametric spaces.

$$\mathrm{val}(x,z) \geq \mathrm{Min}\{\mathrm{val}(x,y), \mathrm{val}(y,z)\}.$$

2.1.2 <u>Theorem</u> (i) (M-Steinhorn) ACVF is C-minimal

(ii) (Haskell, M) Every C-minimal valued field (with the language interpreted naturally) is algebraically closed

"Interpreted naturally": VF – field sort;

$\Gamma$ – value gp; $\mathrm{val}(x,y) := v(x-y)$.

<u>Pf</u>: (i) See 2.0.7. Note $E^C_\alpha$ is an equiv. reln.

The $E_C^\alpha$-class containing $a$ is just

$B_{\geq\alpha}(a) = a + \{x \in K : v(x) \geq \alpha\}$ – a coset of a subgp. of $(K,+)$.

Now $R = B_{\geq 0}(0)$ is the $E_0^C$-class containing $0$.

It is a union of the $E_0^0$-classes $\mathcal{M}+a$ $(a \in R)$, the quotient being $k$, so infinite.

R is in definable bijection with any $B_{\geq\alpha}(0)$, hence with any $B_{\geq\alpha}(a)$. This bijection respects open sub-balls of same radius, so (v) holds. So all $T_{um}$-axioms hold.

Part of



Pf of (ii) The non-singleton balls are a basis of open (in fact, clopen) sets for a topology on VF.

There is a very crude 'cell decomposition theorem', describing definable subsets of $VF^n$ as finite unions of 'cells'. Can prove

- Any 1-variable def. function is locally constant or 'isomorphic'
- Any n-variable def. function is finitely piecewise continuous.
- For definable $X, Y \subseteq M^n$,
  $\dim(X \cup Y) = \mathrm{Max}\{\dim(X), \dim(Y)\}$.

Here, for $X \subseteq M^n$ definable, $\dim(X)$ is the largest $r$ s.t. for some projection $\pi : M^n \to M^r$, $\pi(X)$ has non-empty interior in $M^r$.

Hrushovski-Kazhdan

$F: VF^n \to VF$ (defined on an nhood of $a \in VF^n$)
is differentiable if $\exists$ linear $L: VF^n \to VF$ s.t.

$\forall \gamma \in \Gamma \quad \exists \delta_0 \in \Gamma \quad \forall \delta \geq \delta_0$

if $x = (x_1, \ldots, x_n)$ with $v(x_i) \geq \delta$ for all $i$,

then $v\big(F(a+x) - F(a) - Lx\big) > \delta + \gamma$.

Theorem (H-K) There is definable $Y \subseteq VF^n$
of dimension $< n$ s.t.
$F$ is continuously differentiable on $VF^n \setminus Y$.

Here, if $X \subseteq VF^m$, $\dim X = n$
if $n$ is least s.t. there is a def. map

$$VF^m \to VF^n \times (RV \cup \Gamma)^l$$

with finite fibres.

Original notion of 'C-minimal' was slightly more general, without ultrametric.

C-minimal expansions of ACVF

Consider a complete alg. closed valued field $K$ with archimedean value gp, e.g. $\mathbb{C}_p$.

Consider power series $f = \sum a_\mu T^\mu$

Here $T = (T_1, \ldots, T_n)$, $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n$

$T^\mu = T_1^{\mu_1} T_2^{\mu_2} \ldots T_n^{\mu_n}$, $a_\mu \in R$

Suppose $v(a_\mu) \to \infty$ as $|\mu| \to \infty$

where $|\mu| = |\mu_1| + \ldots + |\mu_n|$.

Then $f$ defines a function $R^n \to R$.

We view $f$ as a function $K^n \to R$, identically zero on $K^n \setminus R^n$.

Tate ring of all such functions $K^n \to R$.

Introduce a new function symbol for each such function, over all $n$.

Obtain some theory $T_{an}$ over language $\mathcal{L}_{an}$.

Lipschitz: Proved QE for a slightly richer language, with more functions (and with a binary function D for divisibility)
Theory $T_{an}^{LR}$

Lipshitz–Robinson: Studied $T_{an}^{LR}$.

It is C-minimal, and hence so is $T_{an}$.

**Analogous notion 'P-minimal' for $\mathbb{Q}_p$.**

Theorem (van den Dries, Haskell, M)
The subanalytic expansion of $\mathbb{Q}_p$ investigated by Denef, van den Dries is P-minimal.

**Real closed valued fields**
**– weakly o-minimal.**

<u>Def</u>: A complete theory $T$ has the <u>independence property</u> if for some $m, n$ there is a formula $\phi(x, y)$ $(l(x) = m,\ l(y) = n)$, some $M \models T$ containing $\{a_i : i \in \omega\} \subseteq M^m$, s.t. for any $S \subset \omega$ there is $b_S \in M^n$ with

$$M \models \phi(a_i, b_S) \text{ iff } i \in S.$$

**A complete theory without indep. prop. is called <u>NIP</u>.**

<u>Fact</u>: If $T$ has IP, then there is a formula $\phi(x, y)$ as above with $m = 1$.

NIP theories **heavily studied recently, by Shelah, Adler, Hrushovski-Pillay, ...**

<u>Prop 2.1.3</u> If $T$ is strongly minimal, o-minimal, C-minimal, P-minimal (or weakly o-minimal) then $T$ is NIP.

<u>Idea of pf in C-minimal case</u>: If $T$ is IP in the bigger language $L^+$, we can suppose it is witnessed by some $\phi(x, y)$ with $x$ a singleton.

Can also suppose $\phi$ is a quantifier-free $L$-formula.

Have $\{a_i : i \in \omega\}$. Can arrange

<u>either</u>: all the $a_i$ lie in different open $\gamma$-subballs of a closed $\gamma$-ball

<u>or</u>: for some $a$, all the $v(a - a_i)$ are distinct.

Then finish by C-minimality.

# §3 Imaginaries in ACVF

Imaginaries in valued fields investigated by Holly, Macintyre – Scowcroft.

**Def:** (i) Complete theory $T$ has <u>elimination</u> of <u>imaginaries</u> (EI) if for all $M \models T$, $n > 0$, $\emptyset$-def equiv rel'n $E_R$ on $M^n$, there is $m > 0$ and $\emptyset$-def. $f_E : M^n \to M^m$ s.t. $\forall x, y \in M^n$, $E(x,y) \Leftrightarrow f(x) = f(y)$.

(ii) An <u>imaginary</u> is an equiv. class of some $\emptyset$-def equiv. rel'n. on some $M^n$.

Can form $M^{eq}$ (in language $\mathcal{L}^{eq}$, with theory $T^{eq}$) by adding, for each $E$ on $M^n$ as above, a <u>sort</u> $M^n/E$ (with appropriate functions identifying it.)

<u>Fact:</u> $Th(M^{eq}) = T^{eq}$ has elimination of imaginaries.

(but definability in $M^{eq}$ is hard to handle).

The idea of EI.

Let $R \subset M^n \times M^t$ be a $\emptyset$-def. relation.
This is really a family of definable sets in $M^t$,
with parameters in $M^n$. For $a \in M^n$

$$R_a := \{ y \in M^t : R(a,y) \} \subset M^t.$$

**Define** equiv. reln. $E_R$ on $M^n$, by:

$$E_R \, x y \iff R_x = R_y \qquad \emptyset\text{-def.}$$

We say $x/E_R$ codes $R_x$, write $x/E_R = \ulcorner R_x \urcorner$.

Here $x/E_R$ is an element of $M^{eq}$.

Given EI, there is $m > 0$ and $\emptyset$-def $f : M^n \to M^m$
s.t. $f(x) = f(y) \iff E_R \, xy \iff R_x = R_y$.
So now $f(x)$, a tuple from $M$, can be viewed
as a code for $R_x$.

Put $Z := \mathrm{Im}(f)$. Can replace $R$
by $R' \subset Z \times M^t$, where

$$R(x,y) \iff R'(f(x), y)$$

$R$ and $R'$ define the same family of sets in $M^t$,
but w.r.t. $R'$, each set has a unique parameter.

<u>NB:</u> $EI$ means: for each imaginary $e$ of $\mathcal{U}$ (the monster model, i.e. $e \in \mathcal{U}^{eq}$)

$$e \in dcl(dcl(e) \cap \mathcal{U}).$$

<u>Weak EI</u>: for each imaginary $e$,

$$e \in dcl(acl(e) \cap \mathcal{U}).$$

<u>Examples 3.2</u> (i) Pure set. Weak $EI$, but not $EI$.

(ii) RCF (or any o-minimal expansion of it).
$EI$ holds, as for any $\emptyset$-def. equiv. reln $E$ on $M^n$, there is $\emptyset$-def $f : M^n \to M^n$ picking out a rep. of each $E$-class.

(iii) ACF has $EI$. Can code a finite set of field elements $\{a_1, \ldots, a_n\}$ by the sequence of coeffs. of $\prod (X - a_i)$.
+: any affine variety has a smallest field of def.

(iv) ACFA has $EI$.

Remark: ACVF does not have EI in the sort K. We cannot code els. of $\Gamma$, $k$ in $K$, e.g. by dimension arguments. (Ex.)

ACVF does not have EI to the sorts $K, k, \Gamma$ (Holly). So add more sorts from $K^{eq}$.

Initial hope: to obtain EI when you add sorts for open and closed balls (false).

Defn. An n-lattice of $K$ is a free rank $n$ $R$-submodule of $K^n$.

An n-lattice will have form $Rv_1 \oplus \dots \oplus Rv_n$ for some $K$-lin. indep $v_1, \dots, v_n \in K^n$.

Let $Z_n$ be set of ordered bases $(z_1, \dots, z_n)$ of $K^n$, so $Z_n \subseteq K^{n^2}$.

Define an equiv. reln $E_{S_n}$ on $Z_n$:

$$(x_1, \dots, x_n) E_{S_n} (y_1, \dots, y_n) \Leftrightarrow \sum R x_i = \sum R y_i.$$

So equivalence classes of $E_{S_n}$ correspond to n-lattices.

Let $S_n$ be set of equiv. classes of $E_{S_n}$ – a set of codes for n-lattices.

Can also view $S_n$ as a coset space.

$GL_n(K)$ acts transitively (by left mult.) on set of ordered bases of $K^n$, hence transitively on set of $n$-lattices.

One of these $n$-lattices is $R^n$ (with standard basis). The stabiliser (in the action of $GL_n(K)$) of $R^n$ is just $GL_n(R)$ — the group of $n \times n$ matrices over $R$ with inverses over $R$.

So by orbit-stabiliser theorem,

$$S_n \longleftrightarrow GL_n(K) / GL_n(R) \qquad \text{(coset space)}$$

We can also identify $S_n$ with $B_n(K) / B_n(R)$,

where $B_n(K)$ is the group of invertible upper-triangular matrices.

$S_1$ is in bijection with $\Gamma$:

$$S_1 = GL_1(K)/GL_1(R) = K^*/U(R) \cong \Gamma.$$

So $S_1$ has a natural group structure.

Note each 1-lattice has form $\gamma R = \{x \in K : v(x) \geq \gamma\}$.

For $n \geq 2$, $GL_n(R) \ntrianglelefteq GL_n(K)$, and there is no natural group structure on $S_n$.

<u>NB</u>: We may view set $\mathbb{B}^{cl}$ of closed balls as a <u>sort</u> in $K^{eq}$.

Define $E$ on $\{(x,y) \in K^2 : x \neq y\}$ by

$$(x,y)\, E\, (x',y') \Leftrightarrow B_{\geq v(x-y)}(x) = B_{\geq v(x'-y')}(x')$$

(Similarly for open balls)

Now a closed ball containing $0$, say $B_{\geq\gamma}(0)$, is the 1-lattice $\gamma R$. A closed ball not containing $0$, say $B_{\geq\gamma}(a)$ (where $v(a) < \gamma$) is a <u>torsor</u> for $\gamma R$.

$$B_{\geq\gamma}(a) = B_{\geq\gamma}(0) + a.$$

Can identify $B = B_{\geq\gamma}(a)$ with an el. of $S_2$.

$\{1\} \times B$ generates an $R$-module $L$ in $S_2$, and

(i) $B = L \cap (\{1\} \times K)$

(ii) $L$ is a 2-lattice.

If $L$ is an $n$-lattice, then

$\mathcal{M}L := \{ax : a \in \mathcal{M},\ x \in L\}$ is an $R$-submodule of $L$.

The quotient $\mathrm{red}(L) := L/\mathcal{M}L$ has structure of an $n$-dim vector space over $R/\mathcal{M} = k$.

In particular, if $L = \gamma R = aR$ ($a \in K$ with $v(a) = \gamma$) is a 1-lattice,

then $\mathrm{red}(L) = \gamma R/\gamma\mathcal{M} = \{\gamma\mathcal{M} + a : v(a) \geq \gamma\}$
$= \{B_{>\gamma}(a) : v(a) \geq \gamma\}$.

Here $\mathrm{red}(L)$ is the set of open sub-balls of $\gamma R$ of the same radius $\gamma$. It is a torsor of $k$, so is strongly minimal.

The set $\bigcup_{\ulcorner L \urcorner \in S_n} \mathrm{red}(L)$ is a uniformly definable family of subsets of $K^n$.

Define $T_n$ to be the set of codes for members of this family.

Can view $T_n$ as a quotient, by $E_{T_n}$, of

$$X_n := \{ (x, y_1, \dots, y_n) \in K^{n(n+1)} : \begin{array}{l} y_1, \dots, y_n \in K^n \text{ are lin. indep.}, \\ x \in Ry_1 \oplus \dots \oplus Ry_n \end{array} \}$$

$(x, y_1, \dots, y_n)\ E_{T_n}\ (x', y_1', \dots, y_n')$ iff

$Ry_1 \oplus \dots \oplus Ry_n = Ry_1' \oplus \dots \oplus Ry_n' = L$, AND

$$x + \mathcal{M}L = x' + \mathcal{M}L.$$

We have $\pi_n : T_n \to S_n$

$$\pi_n \left( (x, y_1, \dots, y_n)/E_{T_n} \right) = (y_1, \dots, y_n)/E_{S_n}.$$

Fibres of $\pi_n$: set of codes for els. of some $\mathrm{red}(L)$.

$T_1 = \{ \ulcorner B_{>v(a)}(a) \urcorner : a \in K^* \}$, and

$$\{ B_{>v(a)} : a \in K^* \} = \{ a(1+\mathcal{M}) : a \in K^* \}$$

$$= K^*/1+\mathcal{M} = RV.$$

So like $S_1$, $T_1$ is a group.

We can view $T_n$ as a collection of coset spaces.

For $m = 1, \ldots, n$, let

$B_{n,m}(k)$ = set of els. of $B_n(k)$ with $m^{\text{th}}$ column $\begin{array}{c@{}l} \begin{pmatrix} 0 \\ a \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} & \begin{array}{l} \\ \\ \\ -m \\ \\ \\ \end{array} \end{array}$

$B_{n,m}(R)$ = set of els of $B_n(R)$ which reduce mod $\mathcal{M}$ to $B_{n,m}(k)$ (a group).

Can show

$$T_n \text{ `='} \bigcup_{m=0}^{n} B_n(K) \big/ B_{n,m}(R)$$

Here $B_{n,0}(R) := B_n(R)$.

Language $\mathcal{L}_{\mathcal{G}}$ ($\mathcal{G}$ for 'geometric sorts')

has sorts $K$, $k$, $\Gamma$, $S_n$, $T_n$ $(n \geq 1)$.

Don't really need $k, \Gamma$ as $k \subset T_1$, $\Gamma = S_1$.

ACVF has QE in reasonable language $\mathcal{L}_{\mathcal{G}}$ with these sorts.

<u>Theorem</u> (Haskell, Hrushovski, M).

ACVF has EI in sorts of $\mathcal{L}_{\mathcal{G}}$.

<u>Remarks</u>

(1) Essentially, we <u>need</u> this level of complexity of sorts.

(2) For every imaginary $e$, there is some $R$-submodule of some $K^n$ with code interdefinable with $e$.

(3) (Mellor) RCVF has EI in same sorts.

(4) (Hrushovski, Martin) $Th(\mathbb{Q}_p)$ has EI in sorts $K$, $S_n$ $(n \geq 1)$.

Application of EI for $\mathbb{Q}_p$ (Hrushovski, Martin)

Let $G$ be a fin. gen. nilpotent group.

$R_n(G) :=$ set of irred complex $n$-dim. characters.

Define $\sim$ on $R_n(G)$ : $\sigma_1 \sim \sigma_2 \Leftrightarrow$

$\Leftrightarrow$ there is linear character $\chi$ with $\sigma_1 = \chi \sigma_2$.

<u>Fact</u> (Lubotsky, Magid 1985).

$$a_n(G) := \left| R_n(G)/_{\sim} \right| \text{ is finite.}$$

Now define $\zeta_{G,p}(t) = \sum_{n=0}^{\infty} a_{p^n} t^n$.

<u>Theorem</u> (Hrushovski, Martin)

$\zeta_{G,p}(t)$ is a rational function.

**Theorem** (Hrushovski, Martin

Let $R = (R_\iota)_{\iota \in \mathbb{N}^r}$ be a definable family of subsets of $\mathbb{Q}_p^N$. Let $E = (E_\iota)_{\iota \in \mathbb{N}^r}$ be a definable family of equivalence relations on $(R_\iota)_{\iota \in \mathbb{N}^r}$. Suppose that for each $\iota \in \mathbb{N}^r$, $|R_\iota / E_\iota|$ is finite.

Put $a_\iota := |R_\iota / E_\iota|$.

Then the power series

$$\sum_{\iota \in \mathbb{N}^r} a_\iota t^\iota \in \mathbb{Q}[[t_1, \ldots, t_r]]$$

is $\mathbb{Q}$-rational.

Other applications for group-theoretic rationality results.

Let $G$ be a fin. gen. torsion-free nilpotent group, $p$ a prime, and let $b_n$ be one of:

(i) number of index $p^n$ subgroups

(ii) number of conjugacy classes of index $p^n$ subgroups

(iii) –

(iv) –

Then $\sum b_n t^n$ is a rational function

(Grunewald, Segal, Smith 1988, using work of Denef on p-adic integrals.

Nice proof by Hrushovski - Martin using $EI$, for $\mathbb{Q}_p$.

# §4 Proof of EI in ACVF

Key issues:

(1) No straightforward independence theory (apart from that arising from algebraic closure, which is as for ACF).

(2) Every element of a ball $B_{\geq\gamma}(b)$ is a centre of it – no canonical centre.

This suggests one should add sorts for balls (i.e. a sort for open balls, and a sort for the collection of closed balls). Not sufficient.

I'll describe aspects of the proof of EI (in the language $L_{\mathcal{G}}$) which have use <u>beyond</u> the proof itself.
(e.g. metastable theories, Hrushovski-Kazhdan)

Often work over a parameter base $C$. I'll assume $C = \mathrm{acl}^{eq}(C)$ just for simplicity (not needed for some statements)

(1) Description of 1-types (in the sort $K$) over $C$. (Applies also to $S_n$, $T_n$, e.g. for $S_n$ due to the 'triangular' presentation

$$S_n \sim B_n(K)/B_n(R)^{-} \quad )$$

Let $a \in K$. Define

$\mathcal{B}_C(a) := \{B : B \text{ a } C\text{-def. ball}, \ a \in B\}$

The balls in $\mathcal{B}_C(a)$ form a chain under inclusion (as for any balls $B_1, B_2$, $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$ or $B_1 \cap B_2 = \emptyset$).

**Define $Loc_C(a) := \bigcap \{B : B \in \mathcal{B}_C(a)\}$**

**(viewed as a subset of $\mathcal{U} \cap K$.)**

If $\mathcal{B}_C(a)$ has a smallest ball, $Loc_C(a)$ equals this ball (viewed as a subset of $K \cap \mathcal{U}$.)

Otherwise, $Loc_C(a)$ is an $\infty$-def. set in $K$, over $C$.

- $b \in K$ is <u>generic</u> on $Loc_C(a)$ if $Loc_C(b) = Loc_C(a)$, i.e. $b$ and $a$ lie in the same $C$-definable balls.

- If $C \subset D$, <u>$a$ is generic in $Loc_C(a)$ over $D$</u> if $Loc_C(a) = Loc_D(a)$. Then write $a \downarrow^g_C D$.

  So view balls as like Zariski closed sets

<u>Lemma</u> (4.2.1) Suppose $C \subset D$. Then

(i) If $B$ is a $C$-definable ball, or the intersection of a chain of $C$-def balls, then $B = Loc_C(a)$ for some $a \in B$.

(ii) If $b$ is generic in $Loc_C(a)$, then $a \equiv_C b$
(so $Loc_C(a)$ has a unique generic type, realised by $a$)

(iii) $tp(a/C)$ has a generic extension over $D$.

(iv) If $a \equiv_C a'$, $a \downarrow^g_C D$, $a' \downarrow^g_C D$, then $a \equiv_D a'$
(stationarity)

(v) If $B$ is a ball, the generic type of $B$ over $\mathcal{U}$ is definable over $\ulcorner B \urcorner$.

**[NB: All this holds in any C-minimal theory in Hrushovski - Kazhdan sense ]**

Pf: (i) Use compactness to choose $a \in B$ outside any C-def. proper sub-balls of B: B is not the union of finitely many proper sub-balls.

(ii) Since def. subsets of K are bool. combs. of C-def. balls, any 1-type over C is determined by C-def. balls.

(iii) As for (i)

(iv) As for (ii)

(v) e.g. suppose $B = B_{\geq\gamma}(a)$ is closed, radius $\gamma$.
The set of open sub-balls of B of radius $\gamma$ ("red(B)") is strongly minimal, so for any formula $\phi(x,y)$ ($x$ a single field variable) there is $n_\phi$ s.t. if $\phi(x,a)$ lies in the union of finitely many els. of red(B), it is in a union of $\leq n_\phi$ els. of red(B); otherwise, $\phi(x,a)$ contains the union of all but $\leq n_\phi$ els. of red(B).

So if p is the generic type of B,

$\phi(x,a) \in p \iff \phi(x,a)$ meets more than $n_\phi$ els. of red(B).

## (2) The <u>stable part</u> of ACVF.

Work over params $C$.

**Def** A $C$-definable set $D$ is <u>stable</u> if it is stably embedded, and is stable as a structure with $C$-definable relations.

Various equivalences, e.g.

TFAE (i) $D$ is stable

(ii) for any formula $\phi(x_1, \ldots, x_n, y)$ which implies $\bigwedge D(x_i)$, $\phi$ is stable (when parsed as $\phi(\bar{x}, y)$)

(iii) If $\lambda > |T| + |C|$, $\lambda = \lambda^{\aleph_0}$ and $B \supseteq C$ with $|B| = \lambda$, there are $\leq \lambda$ 1-types over $B$ which are realized in $D$.

<u>Defn</u>: If $D_1$, $D_2$ are definable sets in $\mathcal{U}$, then $D_2$ is <u>$D_1$-internal</u> if there is finite $F$ s.t. $D_2 \subseteq \mathrm{dcl}(D_1 \cup F)$

(So for some $n$ there is a definable <u>surjection</u> $D_1^n \longrightarrow D_2$. )

Recall: If $s \in S_n$ then red(s) means red($L$) where $L$ is the $n$-lattice coded by s. So red(s) is an $n$-dimensional k vector space.

Lemma If $s \in S_n$ then red(s) is k-internal and stable (viewed as an $\ulcorner s \urcorner$-definable set).

Pf: red (s) $\subseteq$ dcl (k $\cup$ {basis for red (s)}) .

finite

As k is stable, so is red (s) (it has Morley rank n).

**Def**: $VS_{k,C}$ is the many-sorted structure with a sort red (s) for each $s \in S_n \cap dcl(C)$ (i.e. a sort for each C-definable lattice).

Equip $VS_{k,C}$ with all C-definable relations on products of sorts.

So $VS_{k,C}$ is a many-sorted stable structure. Any finite union of sorts has finite Morley rank.

ACVF

<u>Theorem</u> Let $D$ be a $C$-definable subset of $\mathcal{U}$. The following are equivalent.

(i) $D$ is stable

(ii) $D$ is $k$-internal.

(iii) There is no definable surjection from $D$ to an infinite subset of $\Gamma$.

(iv) $D$ is $k$-analysable

(v) $D \subset \mathrm{dcl}(C \cup VS_{k,C})$.

<u>One point in proof:</u> No infinite definable subset $D$ of $K$ satisfies (iii). Indeed, if $D \subset K$ is infinite and definable, it contains a ball, so we may suppose it is a ball. Pick $d \in D$. Let $D = B_{>\gamma}(d)$ (radius $\gamma$) say.

Define $x \longmapsto v(x-d)$. The range contains $\Gamma^{>\gamma}$.

<u>Theorem</u> $VS_{k,C}$ has elimination of imaginaries

<u>Idea of pf:</u> ① $VS_{k,C}$ is closed under duals, tensor products, exterior powers

② We may easily reduce to coding a definable subset of $\mathrm{red}(S)$

k- Vector space $V$ of red(s).

Now (to prove EI for $VS_{k,C}$),

Via an identification $V \leftrightarrow k^n$ (over a basis)

we have a notion of 'Zariski closed' (indep. of basis)

Via QE in ACF, any def set is a b.c. of

Zariski closed sets

Identify $k[X_1, \ldots, X_n]$ with

$$S(V) := k \oplus V^* \oplus \sum_{i \geq 2} \mathrm{Sym}^i (V^*)$$

$X$ is determined by the ideal vanishing in $k[X_1, \ldots, X_n]$

hence by a subspace $U$ of some

$$S^n (V) = k \oplus V^* \oplus \sum_{i=2}^{n} \mathrm{Sym}^i (V^*)$$

Pull $U$ back to a subspace $U'$ of

$$T^n (V) = k \oplus V^* \oplus \sum_{i=2}^{n} \otimes^i (V^*)$$

Reduce to coding a subspace of some red(s)

Using exterior powers, reduce to coding a 1-space

in some red(s)

3) General lemma for EI.

Lemma: Let $\mathcal{U}$ be a suff. sat. multi-sorted structure, with a sort $D$ s.t. $\mathcal{U} \subset (\mathcal{U} \cap D)^{eq}$.

Suppose that for each sort $S$ of $\mathcal{U}$, every 1-variable partial function $f : D \to \cancel{P}S$ is coded in $\mathcal{U}$. Then $Th(\mathcal{U})$ has EI.

Pf: Show by induction on $n$ that each definable $R \subset D^n$ is coded.

For $n=1$, any $R \subset D$ is coded by $id_R$.

For inductive step, we have $R \subset D^{n+1}$ and

$\pi : D^{n+1} \to D$ (to first coord.)

Put $Y := \pi(R)$.

For each $y \in Y$, the fibre $R_y \subseteq D^n$ so (by induction) is coded by some $h(y)$.

By compactness, $h$ is definable, and we may write $Y = Y_1 \dot\cup \dots \dot\cup Y_t$ s.t. $h|_{Y_i}$ is to a specific product of sorts.

Now by assumption, each $h|_{Y_i}$ is coded in $S$, and $(\ulcorner h|_{Y_1} \urcorner, \dots, \ulcorner h|_{Y_t} \urcorner\}$ codes $R$.

4) The proof of EI. Some other ingredients

i) Definable $R$-submodules of $K^n$ are coded in $\mathcal{G}$

ii) Definable functions $\Gamma \to \mathcal{G}$ (and their germs) are coded in $\mathcal{G}$.

iii) Finite sets of tuples from $\mathcal{G}$ are coded in $\mathcal{G}$.

Proposition: Let $B$ be a $C$-definable ball (or intersection of chain of $C$-def balls). Let $f$ be a def. partial function $K \to \mathcal{G}$ s.t. $\mathrm{dom}(f) \supseteq B$, and assume $C = \mathrm{acl}(C \ulcorner f \urcorner) \cap \mathcal{G}$. (e.g. if $\mathrm{dom}(f) = B$ and $C = \mathrm{acl}(\ulcorner f \urcorner) \cap \mathcal{G}$)

Then there is a $C$-def. function $g$ with the same germ on $B$ as $f$.

Idea of pf: First prove a related result for germs of functions on closed balls showing the germ is coded by a code for a def. $R$-module (use (i))

Then approximate $B$ from inside by closed balls, using (ii).

Completion of pf of EI: We code a def. partial fn. $f: K \to \mathcal{G}$.

Show $\ulcorner f \urcorner \in \mathrm{dcl}(\mathrm{dcl}_{\mathcal{G}}(\ulcorner f \urcorner))$, and by (iii) suff. to get $\ulcorner f \urcorner \in \mathrm{dcl}(\mathrm{acl}_{\mathcal{G}}(\ulcorner f \urcorner))$.

Put $C := \mathrm{acl}_{\mathcal{G}}(\ulcorner f \urcorner)$, and

$\Sigma := \{ D \subset \mathrm{dom}(f) : D, f|_D$ are $C$-definable$\}$

If $\bigcup \Sigma = \mathrm{dom}(f)$, we are done by compactness.

So suppose $\bigcup \Sigma \subsetneq \mathrm{dom}(f)$, and find a complete type $p/C$ of realisations of $\mathrm{dom}(f) \setminus \bigcup \Sigma$.

Then $p$ is the generic type of a $C$-def. ball or chain of balls $B$.

There is $C$-def. $g$ with same germ on $B$ as $f$.

Put $X := \{ x \in B : f(x) = g(x) \}$

Then $X$ contains a realisation of $p$.

$X$ is coded in $\mathcal{G}$ (as in one variable) and is $\ulcorner f \urcorner C$-definable, so is $C$-definable.

So $C$ contains all realisations of $p$ (as $p \in S(C)$).

As $g$ is $C$-def, so is $f|_X$, so $X \in \Sigma$.

Contradiction, as $p$ is a type of elt. of $\mathrm{dom}(f) \setminus \Sigma$.

How are modules used in the coding?

An aspect of the coding of finite sets:

Suppose $t_1, \ldots, t_m$ are open balls of radius $\gamma$, and there is $\delta < \gamma$ s.t. if $t_i \neq t_j$, $x \in t_i$, $y \in t_j$, then $v(x-y) = \delta$.

Put $F = \{t_1, \ldots, t_m\}$.

Let $T := t_1 \cup \cdots \cup t_m$.

Define

one variable

$$J^F = \{ f \in K[X] : \deg(f) \leq m,\ \forall x \in T \left( v(f(x)) > (m-1)\delta + \gamma \right) \}$$

Then $J^F$ can be viewed as a definable $R$-submodule of $K^{m+1}$, so is coded in $\mathcal{G}$.

Also $(\ulcorner J^F \urcorner, \gamma, \delta)$ is a code for $F$.

**The point:** If $f \in K[X]$ is monic of degree $m$, then

$$f \in J^F \iff f \text{ has a root in each } t_i.$$