§5 Stable Domination

Idea: In ACVF, over a base C, we have a stable structure $VS_{k,C}$. What role does it play in understanding types? We consider certain types which are governed by a 'trace' in $VS_{k,C}$. Applications in §6, e.g. to definable groups in ACVF (Hrushovski).

Material from Part I of:

D. Haskell, E. Hrushovski, HD Macpherson, Stable domination and independence in algebraically closed valued fields, LNL, CUP.

Recall: A complete type p over $M \models T$ is <u>definable</u> over $C \subset M$ if for each formula $\phi(x, y)$ there is a formula (over C) $d_p x\, \phi(x,y)$ (just in the y variables) such that for any a,

$$\phi(x,a) \in p \quad \text{iff} \quad d_p x\, \phi(x,a).$$

So $d_p x\, \phi(x,y)$ picks out the parameters a.

Now suppose $\mathcal{U} \models T$ is sufficiently saturated, homogeneous. $C \subset \mathcal{U}$ is small. Then $Aut(\mathcal{U}/C)$ acts on the space of types over $\mathcal{U}$: for $g \in Aut(\mathcal{U}/C)$,

$$\phi(x,a) \in gp \iff \phi(x, g^{-1}a) \in p.$$

We say $p \in S(\mathcal{U})$ is <u>$Aut(\mathcal{U}/C)$-invariant</u> (or <u>C-invariant</u>, or <u>invariant</u>) if p is fixed by $Aut(\mathcal{U}/C)$.

$p \in Aut(\mathcal{U}/C)$ is C-invariant

$$\iff$$

p does not split over C, i.e.

$$\left(\begin{array}{c} \text{for any } a_1, a_2 \in \mathcal{U}, \quad \text{if } a_1 \equiv_C a_2, \\ \text{then} \\ \phi(x,a_1) \in p \iff \phi(x,a_2) \in p. \end{array} \right)$$

<u>Examples</u>

(i) If $\mathcal{U} \models T_{DLO}$, then "$x > \mathcal{U}$" and "$x < \mathcal{U}$" determine $\mathrm{Aut}(\mathcal{U}/\emptyset)$-invariant 1-types (and are the only such).

(ii) If $\mathcal{U} \models$ Theory (random graph) then "x is joined to $\mathcal{U}$", or "x is not joined to $\mathcal{U}$" are $\mathrm{Aut}(\mathcal{U}/\emptyset)$-invariant (and are only such).

(iii) (ACVF) The generic type (over $\mathcal{U}$) of a C-definable ball, or of a chain of C-def. balls, is $\mathrm{Aut}(\mathcal{U}/C)$-invariant.

(iv) If $\mathcal{U}$ is o-minimal (or just weakly o-minimal) then every type over C has an $\mathrm{Aut}(\mathcal{U}/C)$-invariant extension (for 1-types, at most two such).

<u>Facts:</u> (1) Every C-definable type is $\mathrm{Aut}(\mathcal{U}/C)$-invariant

(2) Converse false: e.g., in ACVF, the generic type of a chain of balls with no least element (i.e. of the intersection of the balls) is C-invariant, not definable.

(3) (Assume T is NIP) Every $\mathrm{Aut}(\mathcal{U}/C)$-invariant type is <u>Borel-definable</u> over C (Hrushovski-Pillay)

(4) (Assume T NIP) If every 1-type (in $\mathcal{U}$) over every $C = \mathrm{acl}^{eq}(C)$ has an $\mathrm{Aut}(\mathcal{U}/C)$-invariant extension, then the same holds for all types (in $\mathcal{U}^{eq}$) (H-P)

(5) Every definable C-invariant type is C-definable.

(6) Invariant types have Morley sequences, which are indiscernible.

(7) (By (2) and (4)): In ACVF, if $C = \mathrm{acl}(C)$ then <u>every</u> type over C has a C-invariant extension

The structure St_C:

For convenience, assume T has EI, and assume $C = acl(C)$ (not needed in all assertions).

Defn: St_C is the many-sorted structure with a sort for each C-definable stable set, and with the C-definable relations on products of sorts. (cf: $VS_{k,C}$).

If $A \subset \mathcal{U}$, $St_C(A) := dcl(CA) \cap St_C$ (often viewed as an infinite tuple)

$$St_C(a) := dcl(Ca) \cap St_C$$

NB: 1) The elements of C are singleton sorts of St_C, and lie in any $St_C(a)$

2) In ACVF, St_C is "essentially" $VS_{k,C}$

If $C \models ACVF$, then St_C is "essentially" k.

<u>Defn</u> 5.2.2 : $tp(a/C)$ is stably dominated if, for any b,

if $St_C(a) \underset{C}{\downarrow} St_C(b)$ (in the stable structure St_C)

then $tp(b/St_C(a)) \vDash tp(b/Ca)$

Formally this means:

for any b', if $b' \underset{St_C(a)}{\equiv} b$ then $b' \underset{Ca}{\equiv} b$ (*)

<u>Ex</u>: By automorphism argument (using that sorts in St_C are stably embedded)

(*) is equivalent to :

for any a', if $a' \underset{St_C(b)}{\equiv} a$, then $a' \underset{Cb}{\equiv} a$

i.e. $tp(a/St_C(b)) \vDash tp(a/Cb)$

2. Another presentation:

View $St_C(a)$ as an infinite tuple (wrt some enumeration) so we have a map $f : a \mapsto St_C(a)$ on $tp(a/C)$.

Let D be any definable set (in sort containing a) and $d := \ulcorner D \urcorner$.

Say a fibre X of f is generic if for some/any $a \in X$,

$$St_C(a) \underset{C}{\downarrow} St_C(d).$$

Then $tp(a/C)$ is stably dominated iff, for each D, either D contains all generic fibres of f, or D is disjoint from all generic fibres of f.

(cf. compact domination).

3. $tp(a/C)$ is stably dominated iff, for any b with $St_C(a) \underset{C}{\downarrow} St_C(b)$

$tp(a/St_C(a))$ has a unique extension over $St_C(a)b$

NB: Without the assumption $C = acl(C)$, we just work with the first condition (the above equivalences use stationarity in St_C)

<u>Basic Example</u> (ACVF). Let p be the generic type over C of $R = B_{\geq 0}(1)$ (or of any closed ball).

For simplicity, assume C is a model, so St_C is essentially just k.

If $a \models p$, then $res(a)$ is transcendental in k over $acl(C) \cap k$ (and essentially, $res(a)$ $\underset{\equiv}{is}$ $St_C(a)$).

For any $B \supset C$, put $res(B) := dcl(B) \cap k$

(essentially this is $St_C(B)$).

There is a <u>unique</u> complete type p' over B s.t. p' contains the formula '$x \in R$' and implies (*)

$$res(x) \underset{C}{\downarrow} res(B).$$

Thus, p is stably dominated. More generally, for any C-definable closed ball b, the generic type over C of b is stably dominated.

Reason for (*): Otherwise, there is a formula $\phi(x)$ over B which meets each open ball $\mathcal{M} + a$ (for generic $a \in R$) in a proper non-empty set. This contradicts that $\phi(x)$ defines a b.c. of balls.

Theorem (5.2.3) Let $C = \mathrm{acl}(C)$.

('Definable') (i) If p is stably dominated, then p has a C-definable (so $\mathrm{Aut}(\mathcal{U}/C)$-invariant) extension over $\mathcal{U}$. In fact, p has a unique $\mathrm{Aut}(\mathcal{U}/C)$-invariant extension.

('Ascent') (ii) If $p \in S(\mathcal{U})$ is $\mathrm{Aut}(\mathcal{U}/C)$-invariant, and $C \subset B \subset \mathcal{U}$, then

$p|C$ stably dom. $\Longrightarrow$ $p|B$ stably dom.

('Descent') (iii) If $p, q \in S(\mathcal{U})$ are $\mathrm{Aut}(\mathcal{U}/C)$-invariant, and $b \models q|C$, then

$p|Cb$ stably dom. $\Longrightarrow$ $p|C$ stably dom.

(Q: Do we need invariance of q?)

('transitivity') (iv) If $tp(a/C)$ and $tp(b/Ca)$ are stably dominated, so is $tp(ab/C)$.

The point in 5.23 (i):

Lemma: Suppose C is small, and $r(x,y)$ is a type over $\mathcal{U}$ in possibly infinite tuples x, y. Let $q(x)$ be the restriction of r to the x variables.

Suppose q is C-definable, and $r(x,y)$ is the unique extension over $\mathcal{U}$ of $r(x,y)|C \cup q(x)$.

Then $r(x,y)$ is C-definable.

Application of lemma, to prove definability of $p(y) \in S(\mathcal{U})$, with $p|C$ stably dominated:

Choose $a \models p$ with $St_C(a) \underset{C}{\downarrow} St_C(\mathcal{U})$.

Let $r(x,y) = tp(St_C(a), a / \mathcal{U})$

$q(x) = tp(St_C(a) / \mathcal{U})$

Since $q(x)$ is definable over C in the stable structure St_C, by the lemma r is C-definable, and hence so is p.

Strong codes for germs:

Suppose $p \in S(\mathcal{U})$ is C-definable, and f_a is an a-definable function, defined on realisations of p.

Define $\sim$: $a \sim a'$ iff for each $x \models p$,

$$f_a(x) = f_{a'}(x)$$

i.e. $a \sim a'$ iff '$f_a(x) = f_{a'}(x)$' $\in p$.

Then $\sim$ is a C-definable equiv. reln.

The $\sim$-class of a is an imaginary, called the p-germ of a.

Also, if f, g are defined on p, say f, g have the same p-germ if

'$f(x) = g(x)$' $\in p$.

Def The p-germ e of f_a is strong over C if there is a Ce-definable function g defined on p and with the same p-germ as f_a.

Fact: In a stable theory, any germ of a definable function on a definable global type is strong.

Theorem 5.24: Let $C = \mathrm{acl}(C)$ and $p \in S(\mathcal{U})$ with $p|C$ stably dominated. Let f_a be a definable function defined on realisations of p. Then:

(i) the p-germ of f_a is strong over C;

(ii) suppose $f(b) \in St_{Cb}$ for $b \models p$.

Then the p-germ of f is in St_C.

NB: For (i), it suffices to show:

Let e be the p-germ of f.

Then for any $a \equiv_{Ce} a'$,

if $b \models p|Ca$ and $b \models p|Ca'$, $\quad (*)$

then $f_a(b) = f_{a'}(b)$.

(Given this, $f_a(b) \in \mathrm{dcl}(Ceb)$. Then use compactness.)

In stable case, $(*)$ is easy:

choose a'' with $a'' \underset{Ce}{\downarrow} a a' b$.

Here it is more complicated.

Generalisation of stable domination,
considered recently by Hrushovski, Pillay.

<u>Defn:</u> A C-definable type $p \in S(\mathcal{U})$ is <u>generically stable</u> if it is finitely satisfiable in any small model $M \supset C$.

<u>NB:</u> $p|C$ stably dominated (and p $\mathrm{Aut}(\mathcal{U}/C)$-inv.)
$\Rightarrow$ p generically stable.

Model Theory of Valued Fields

Dugald Macpherson (Leeds)

La Roche, April 2008

I Basics of ACVF

II C-minimality and variants

III Elimination of Imaginaries

IV Proof of EI

V Stable Domination

VI Stable domination in ACVF, generically metastable groups

§6 Stable domination in ACVF, metastable theories

Work in sorts $\mathcal{G}$ (to ensure EI)

Def 6.1.1 Let $C = \mathrm{acl}(C)$, $p \in S(C)$

Then $p \perp \Gamma$ if, for any $\mathrm{Aut}(\mathcal{U}/C)$-invariant extension p' of p over $\mathcal{U}$, and any model $M \geq C$ (M in $\mathcal{U}$),

if $a \models p'|M$ then

$$\mathrm{dcl}_\Gamma(Ma) = \mathrm{dcl}_\Gamma(M) \qquad (*)$$

(Here $\mathrm{dcl}_\Gamma(X)$ means $\mathrm{dcl}(X) \cap \Gamma$)

NB: - Equivalent to the defn given in HHM (which used $\downarrow^g$)

- $(*)$ means just that $\mathrm{tp}(a/M)$ has a unique extension over Γ (by o-minimality of Γ)

Theorem 6.1.2 (ACVF). Let $C = \mathrm{acl}(C)$, $p \in S(C)$

The following are equivalent

(i) $p \perp \Gamma$

(ii) p is stably dominated.

NB: The generic type of a closed ball is stably dominated. The generic type of an open ball (or chain of balls with no least el.) is not.

e.g: If B is the open ball, and M is a model containing $\ulcorner B \urcorner$, then M contains a field element b of B. Now a field element a generic in B over M adds a new el. $v(b-a) \in \mathrm{dcl}_\Gamma(Ma) \setminus \mathrm{dcl}_\Gamma(M)$.

(for if $v(b-a) = \gamma \in \Gamma(M)$, then a is in the M-definable proper sub-ball $B_{\geq\gamma}(b)$ of B, contradicting that a is generic in B.)

Theorem (Baur) 6.1.5:

Let $(F, v) < (L, w)$ be an extension of valued fields, with F maximally complete. Let U be a finite dim. F-subspace of L. Then U has a basis $u_1, \dots, u_n$ s.t. for any $a_1, \dots, a_n \in F$,

$$v(a_1 u_1 + \dots + a_n u_n) = \text{Min}\{v(a_i u_i) : 1 \leq i \leq n\}.$$

Theorem 6.1.7 Let M be a maximally complete model of ACVF. Then for any a (in the sorts $\mathcal{G}$),

$$tp(a / M \cup dcl_\Gamma(Ma)) \text{ is stably dominated.}$$

Key idea: Working over M, this gives a *-definable function $f = (f_i)$ to Γ with stably dominated fibres.

Special case of Thm 6.7

<u>Prop:</u> Let $M \models ACVF$ be maximally complete and a be a sequence of field elements such that $dcl_\Gamma(M \cup a) = dcl_\Gamma(M)$.

Then $tp(a/M)$ is stably dominated.

<u>Idea of Pf:</u> Let $A := \widetilde{M(a)}$, $B \models ACVF$ with $M \leq B$,

and suppose $k(A)$ and $k(B)$ are linearly disjoint over $k(M)$ (i.e. $St_M(A) \underset{M}{\downarrow} St_M(B)$).

Show that if $A' \equiv_C A$ and $k(A'), k(B)$ are lin. disj. over M,

then $A \underset{B}{\equiv} A'$. $(\ast)$

Given $\bar{b}$ from B, we may write

$\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{k} d_j b_j'$ s.t.

(i) $tp(d_1, \ldots, d_k/M)$ depends just on $tp(a_1, \ldots, a_n/M)$

(ii) $tp(b_1', \ldots, b_k'/M)$ -- -- $tp(b_1, \ldots, b_n/M)$

(iii) $v(\sum a_i b_i) = v(\sum_{i=1}^{k} d_j b_j') = \operatorname{Min}\{v(d_j) + v(b_j')\}$.

Thus, if $(a_1, \ldots, a_n) \equiv_M (a_1', \ldots, a_n')$, then

$v(\sum a_i b_i) = v(\sum a_i' b_i)$. This suffices for $(\ast)$,

(as $v(d_j) + v(b_j') = v(d_j') + v(b_j')$) by QE.

Metastable theories. Recent preprint

E. Hrushovski, 'Definable groups, metastable theories'.

on definable groups in ACVF.

$*$-definable set: an ∞-definable set in (possibly) infinitely many variables. View it (e.g. interpreted in $\mathcal{U}$) as an inverse limit of definable sets.

<u>Defn:</u> Assume T complete multi-sorted theory, with a privileged sort Γ. Say T is <u>metastable</u> if:

(i) Γ stably embedded

(ii) no infinite definable subset of Γ^{eq} is stable

(iii) any type over any $C = acl^{eq}(C)$ has a C-invariant extension

(iv) for any partial type p over a base C_0, there is $C_1 \supseteq C_0$ s.t. if $a \models p$, then $tp(a/C_1 \cup dcl_\Gamma(C_1 a))$ is stably dominated.

<u>Remarks</u>: (1) Any o-minimal (or weakly o-minimal) theory is metastable, for stupid reasons.

(2) Condition (iii) is there to ensure 'Descent' works (i.e. decreasing the base preserves stable domination).

(3) ACVF is metastable, with Γ as value gp.

(4) Also, $\mathbb{C}((t))$ and DCVF are metastable.

(5) Sometimes, Hrushovski adds conditions (FD) or its strengthening $(FD)_\omega$.

(FD): – Γ is o-minimal

– for any definable set D, there is an upper bound on the Morley rank of definable images of D in St_C

– similar upper bound on o-min. dimension of definable images in Γ^{eq}.

Consider groups G which are definable (or ∞-definable, or $*$-definable) in a suff. sat. $\mathcal{U} \models T$ with EI

Want notion of 'generic type' of G.

Then a "generically metastable" group is one with a stably dominated generic type.

May be simplest to think of a generically metastable group as one with a stably dominated translation-invariant global type.

<u>Def</u>: Assume $p \in S(\mathcal{U})$ is a C-definable type of elements of G.

If $a \in G$ then p has translates pa, ap which are definable over $C \cup \{a\}$

p is <u>left-generic</u> if, for any $B = acl(B)$ over which p is defined, any right translate pb is B-definable.

<u>right-generic</u> defined similarly.

<u>Fact</u>: A stably dominated left-generic is right generic.

So just say "stably dominated generic type".

There's a notion of 'symmetric' left-generic.
It will also be right generic.
The type at ∞ of $(\mathbb{R},<,+)$ is left and right generic, but not symmetric.

Stably dominated types are symmetric left-gen.

w.r.t. symmetric left-generics, the theory is nice:

- any two symmetric left-generics differ by a left translation
- G has an ∞-def. subgp G^0 of bounded index with a unique symmetric left generic.

Def: G is <u>generically metastable</u> if it has a stably dominated (left) generic

Examples: In ACVF
$(R,+)$ is gen. metastable

$(K,+)$ is not.

Theorem 6.2.4 (Not under 'metastable theory' assumptions).

Let G be a definable group with a translation-invariant stably dominated type p (stably dom. over C).

(i) There is a $*$-definable (over C) stable group $\mathfrak{g}$ and a $*$-definable (over C) gp. hom. $\phi : G \to \mathfrak{g}$ s.t. p is stably dominated over C by ϕ:

i.e. if $a \models p$ and b is any tuple, (with $\phi(a) \downarrow_C St_C(b)$)

$tp(b / C \cup \phi(a)) \models tp(b / C \cup a)$.

(ii) In ACVF, if $C = acl(C)$, then ϕ, $\mathfrak{g}$ can be chosen to be definable.

(iii) Given any other homomorphism $\phi' : G \to \mathfrak{g}'$ with $\mathfrak{g}'$ stable there is $\psi : \mathfrak{g} \to \mathfrak{g}'$ s.t.

$$\phi' = \psi \circ \phi.$$

Example: (a) $G = (R, +)$ $\quad \mathfrak{g} = (k, +)$

(b) $G = SL_n(R)$ $\quad \mathfrak{g} = SL_n(k)$

<u>Pf (sketch)</u>: For a realising p, let

$\theta(a)$ enumerate $St_C(a)$.

For each such a, define f_a on realisations of p,

by $$f_a(b) = \theta(ab).$$

As $f_a(b) \in St_C$, the p-germ e of f_a is in St_C.

As e is Ca-definable, $e \in \theta(a)$.

As e is a <u>strong</u> germ, there is

$f'_{\theta(a)}$, defined over $\theta(a)$, with same p-germ as f_a.

Let $b \models p | Ca$, $d = f_a(b)$.

As St_C is stably embedded, and $d \in St_C$,

$tp(\theta(a), d / C\,\theta(b)) \models tp(\theta(a), d / C\,b)$

(NB: $\theta(a), d$ are in St_C)

So as $d \in dcl(C, \theta(a), b)$, $d \in dcl(C, \theta(a), \theta(b))$

So have a C-definable $*$-function F

with $d = F(\theta(a), \theta(b))$

i.e. $\theta(ab) = F(\theta(a), \theta(b))$

F is generically assoc., etc, so by a group chunk theorem, is the restriction of a group op. ($*$-def.)

θ is generically a homom., so is restriction of a gp. hom.

Other results on metastable groups

Theorem Let G be a generically metastable gp definable in $K \models ACVF$. Then there is an algebraic group H over K and a def homo $f : G \longrightarrow H(K)$ s.t.
$\ker(f)$ is 'boundedly imaginary', i.e. there is no definable map from $\ker(f)$ to an unbounded subset of Γ.

Theorem Let H be an affine algebraic group, G a generically metastable definable subgroup. Then for some algebraic group scheme H_1 over R, G is isomorphic to $H_1(R)$.

Thm: Let T be a metastable theory with FD_ω, and let A be a definable abelian gp. Then there is a definable group Λ in Γ^{eq} and a definable hom. $\lambda : A \to \Lambda$ s.t. $\mathrm{Ker}(\lambda)$ is "limit metastable" i.e. is a direct limit (wrt a $*$-definable directed system) of metastable groups.

NB: $(K, +)$ is limit metastable.