Lecture notes on a course about

Model Theory of Valued Fields

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1 p-adic numbers, definition and motivation

Let p be a prime number.

Definition. The ring of p adic integers is the ring of series of the form:

$$\sum_{i\in\mathbb{N}} a_i \cdot p^i$$

where $a_i \in \{0, \dots, p-1\}$ for all i.
The sum of two p-adic integers is the sum with remainder. Define inductively the sum

$$\sum_{i\in\mathbb{N}} s_i \cdot p^i = \sum_{i\in\mathbb{N}} a_i \cdot p^i + \sum_{i\in\mathbb{N}} b_i \cdot p^i$$

by:

> $r_{-1} := 0$.
>
> $s_i :=$ the rest of $a_i + b_i + r_{i-1}$ mod p, $r_i :=$ the quotient of $a_i + b_i + r_{i-1}$ by p for all i.

The product of two p-adic integers is the product with remainder. Define inductively the product

$$\sum_{i\in\mathbb{N}} t_i \cdot p^i = \sum_{i\in\mathbb{N}} a_i \cdot p^i \cdot \sum_{i\in\mathbb{N}} b_i \cdot p^i$$

by:

$$c_i := \sum_{k\leqslant i} a_k \cdot b_{i-k}, \quad \forall i$$

> $r_{-1} := 0$.
>
> $t_i :=$ the rest of $c_i + r_{i-1}$ mod p, $r_i :=$ the quotient of $c_i + r_{i-1}$ by p for all i.

Remark. $\mathbb{N}$ is a subring of $\mathbb{Z}_p$: it corresponds to the subring of finite series (i.e. series whose terms eventualy become 0).

Examples in $\mathbb{Z}_3$

$$-1 = 2 + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + 2 \cdot 3^4 + \dots$$

Thus, we can identify $\mathbb{Z}$ with $(\mathbb{N} \cup -1 \cdot \mathbb{N}) \subseteq \mathbb{Z}_p$.

$$1 = 2 \cdot (2 + 1 \cdot 3 + 1 \cdot 3^2 + 1 \cdot 3^3 + 1 \cdot 3^4 + \dots)$$

Hence, 2 is invertible.

Fact. $\mathbb{Z}_p$ is a domain. An element

$$\sum_{i \in \mathbb{N}} a_i \cdot p^i \in \mathbb{Z}_p$$

is invertible iff $a_0 \neq 0$.

Remark. Hence, $\forall x \in \mathbb{Z}_p$, there is some $n \in \mathbb{N}$ such that $x = p^n \cdot u$, u invertible.

Definition. $\mathbb{Q}_p$, the field of p-adic numbers, is the quotient field of $\mathbb{Z}_p$.
By the above remark, for $x \in \mathbb{Q}_p$, there is some $N \in \mathbb{Z}$ such that

$$x = p^N \cdot \underbrace{\sum_{i \in \mathbb{N}} a_i \cdot p^i}_{\text{unit of } \mathbb{Z}_p}, \quad a_0 \neq 0$$

Hence, $\mathbb{Q}_p$ can be seen as the ring of all series of the form

$$\sum_{i \geqslant N} a_i \cdot p^i$$

where $a_i \in \{0, \dots, p-1\}$ for all i and $N \in \mathbb{Z}$.

Definition. One defines a valuation $v_p : \mathbb{Q}_p^\times \to \mathbb{Z}$ by:

$$v_p(\sum_{i \geqslant N} a_i \cdot p^i) := N, \quad \text{if } a_N \neq 0$$

It satisfies the usual properties of a valuation, namely:

(A1) $v_p(1) = 0$

(A2) $v_p(xy) = v_p(x) + v_p(y)$

(A3) $v_p(x + y) \geqslant min(v_p(x), v_p(y))$

(A3') In fact, $v_p(x) \neq v_p(y) \Rightarrow v_p(x + y) = min(v_p(x), v_p(y))$

Definition. One defines the p-adic value $|\ |_p : \mathbb{Q}_p^\times \to \mathbb{R}$ by $|x|_p := p^{-v_p(x)}$. It satisfies:

(N1) $|1|_p = 1$

(N2) $|xy|_p = |x|_p \cdot |y|_p$

(N3) $|x+y|_p \leqslant max(|x|_p, |y|_p)$

(N3') $|x|_p \neq |y|_p \Rightarrow |x+y|_p = max(|x|_p, |y|_p)$

Theorem 1. *$(\mathbb{Q}_p, |\ |_p)$ is the Cauchy completion of $(\mathbb{Q}, |\ |_p)$.*

Theorem 2 (Hasse-Minkowski)**.** *Let $\bar{a}, a \in \mathbb{Z} \setminus \{0\}$, and consider the equation*

$$a_1 X_1^2 + \cdots + a_n X_n^2 = a$$

i) there is a solution in $\mathbb{Q}$ $\Leftrightarrow$ there are solution in $\mathbb{R}$ and all the $\mathbb{Q}_p$.

ii) Assume $\forall \bar{x} \in \mathbb{Q},\ \exists \bar{y} \in \mathbb{Z},\ \sum a_i(x_i - y_i)^2 < 1$. There is a solution in $\mathbb{Z}$ $\Leftrightarrow$ there are solution in $\mathbb{R}$ and all the $\mathbb{Z}_p$.

Theorem 3 (Hensel's lemma)**.** *Let $f(X) \in \mathbb{Z}_p[X]$ and $a \in \mathbb{Z}_p$ be such that $|f(a)|_p < 1$ and $|f'(a)|_p = 1$, then there exists $x \in \mathbb{Z}_p$ such that $f(x) = 0$ and $|x - a|_p < 1$.*

Some notations and remarks

- $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid v_p(x) \geqslant 0\}$ is a valuation ring with maximal ideal $p\mathbb{Z}_p$.
- $U_p := \{\text{units of } \mathbb{Z}_p\} = \{x \in \mathbb{Q}_p \mid v_p(x) = 0\}$.
  One has $\mathbb{Q}_p^\times / U_p \cong p^{\mathbb{Z}} \cong \mathbb{Z}$, moreover, the composed map $\mathbb{Q}_p^\times \to p^{\mathbb{Z}} \to \mathbb{Z}$ is v_p.
- $v_p(x^{-1}) = -v_p(x)$. Hence, $\forall x \in \mathbb{Q}_p$, $x \in \mathbb{Z}_p$ or $x^{-1} \in \mathbb{Z}_p$. Thus, $\forall x, y \in \mathbb{Z}_p$, $x|y$ or $y|x$, and one has $x|y \Leftrightarrow v_p(x) \leqslant v_p(y)$, since $x = p^{v_p(x)} \cdot u$ and $y = p^{v_p(y)} \cdot u'$ for some units u and u'.
- One define a section $s_p : \mathbb{Z} \to \mathbb{Q}_p^\times$ of v_p through $s_p(k) := p^k$.
- The residue field of v_p is $\mathbb{Z}_p / p\mathbb{Z}_p \cong \mathbb{F}_p$. One has $U_p/(1 + p\mathbb{Z}_p) \cong \mathbb{F}_p^\times$.
- One defines the angular component map $ac_p : \mathbb{Q}_p^\times \to \mathbb{F}_p^\times$ through

$$ac_p(\sum_{i \geqslant N} a_i \cdot p^i) := a_N \text{ , provided } N = v_p(\sum_{i \geqslant N} a_i \cdot p^i).$$

Remark. One has a strong parallel between the p-adic numbers and the formal power series, as explicited below:

p-adic numbers $\mathbb{Q}_p$	Formal power series $k((T))$
p	T
$\mathbb{Z}_p$	$k[[T]]$
v_p	ord_T
$\mid\ \mid_p$, Cauchy completion...	$\mid\ \mid_T = 2^{-ord_T}$, Cauchy completion...
$\mathbb{Z}_p/(p) \cong \mathbb{F}_p$	$k[[T]]/(T) \cong k$
s_p	s_T
ac_p	ac_T

Theorem 4 (Ax-Kochen-Ershov)**.** *For any non principal ultrafilter $\mathcal{U}$ on the prime numbers $\mathbb{P}$,*

$$\prod_{p\in\mathbb{P}} \mathbb{Q}_p/\mathcal{U} \equiv \prod_{p\in\mathbb{P}} \mathbb{F}_p((T))/\mathcal{U}$$

Consider the formula: σ_4: " Every form of degree 4 in 17 variables has a non-trivial zero."

Fact (Tsen-Lang)**.** σ_4 is true in $\mathbb{F}_p((T))$ for all primes p.

Remark. Artin's conjecture implies σ_4 is true in $\mathbb{Q}_p$ for any p.
It follows from theorem 4 above that σ_4 holds in $\mathbb{Q}_p$ for almost all p. However, Terjanian proved that this is the best possible (hence Artin's conjecture does not fully hold).

2 Modell theory of valued fields

Some formalisations of valuations Let K be a field. We can consider a valuation on K in the following languages:

- (K, v, Γ), where Γ is an ordered abelian group, and $v : K^\times \to \Gamma$ a map satisfying (A1)(A2)(A3).
- (K, V), where V is a valuation ring, that is:

  $$\forall x \in K, x \in V \text{ or } x^{-1} \in V$$

  Hence,

  $$\forall x, y \in V,\ x|y \text{ or } y|x$$

  and there is a well defined total ordering $\leqslant_{div}$ on $K^\times/V^\times$ given by:

  $$\bar{x} \leqslant_{div} \bar{y} :\Leftrightarrow \frac{y}{x} \in V$$

Given a valuation map v, define $V_v := \{x \in K \mid v(x) \geqslant 0\}$. One has $V_v^\times = \{x \in K \mid v(x) = 0\}$.

Reciprocaly, given V a valuation ring, define $\Gamma_V := (K^\times/V^\times, \leqslant_{div})$ and let $v_V : K^\times \to \Gamma$ be the quotient map.

- (K, v, Γ, k) with Γ, v as above, and a new sort k for the residue field (i.e the quotient of V by its unique maximal ideal $\mathcal{M}_V := \{x \in K \mid v(x) > 0\}$).
- (K, v, Γ, k, s), where $s : \Gamma \to K^\times$ is a cross section (that is, a section of v).
- (K, v, Γ, k, ac), where $ac : K^\times \to k^\times$ is an angular component map (that is, ac is multiplicative, and coincide with the residue map on $V^\times$)

Remark. An $\aleph_1$-saturated valued field allways has a cross section. A valued field which has a cross section (e.g. an $\aleph_1$-saturated valued field) also have an angular component map. However, not every valued field has an angular component map.

Notation Let (K, v) be a valued field, we let:
$valK$ be its valuation group
V_K its valuation ring, and
$resK$ be the residue field of V_K.

Definition. Suppose $(K, v) \subseteq (L, v)$, then:

$$valK \subseteq valL$$

$$V_K = V_L \cap K$$

$$resK \subseteq resL$$

Call (L, v) an immediate extension of (K, v) if:

$$valK = val_L,\ resK = resL$$

Lemma. *$(\mathbb{Q}_p, v_p)$ has no immediate extension.*

Proof. Let $(\mathbb{Q}_p, v_p) \subseteq (L, v)$ be such that $valK = val_L = \mathbb{Z}$ and $resK = resL = \mathbb{F}_p$.
Let $x \in L$. We show that x is the limit of a Cauchy sequence of elements of $\mathbb{Q}_p$. Whence, since $\mathbb{Q}_p$ is complete, $x \in \mathbb{Q}_p$. By assumption, $v(x) \in \mathbb{Z}$, so up to replacing x by $x \cdot p^{-v(x)}$, we can w.l.o.g. assume that $v(x) = 0$.
We construct $(n_i)_{i\in\mathbb{N}} \subseteq \mathbb{N}$, increasing, and $(a_i)_{i\in\mathbb{N}} \subseteq \{0, \ldots, p-1\} \subseteq \mathbb{Z}_p$ by induction.

Let $\mathcal{M}$ be the maximal ideal of V_L. Remind that by assumption, $resL = V_L/\mathcal{M} = \mathbb{F}_p$. Let $a_0 \in \{0,\dots,p-1\}$ be such that $x \equiv a_0 \ mod(\mathcal{M})$. With this, $x - a_0 \in \mathcal{M}$. If $x - a_0 = 0$, we are done, hence we can assume that $v(x-a_0)$ is defined[1] and then $v(x-a_0) > 0$. Set $n_0 := 0$.
Let $j > 0$, suppose $(n_i)_{i<j}$ and $(a_i)_{i<j}$ already have been constructed, such that $v(x - \sum_{i<j} a_i \cdot p^{n_i}) > n_{j-1}$ and $n_0 < n_1 < \cdots < n_{j-1}$.
Set $n_j := v(x - \sum_{i<j} a_i \cdot p^{n_i}) \in \mathbb{N}$.
We have $v(p^{-n_j} \cdot (x - \sum_{i<j} a_i \cdot p^{n_i})) = 0$. By the above argument, either $x = \sum_{i<j} a_i \cdot p^{n_i}$ and we are done, or we can take $a_j \in \{0,\dots,p-1\}$ such that

$$v\big(p^{-n_j} \cdot (x - \sum_{i<j} a_i \cdot p^{n_i}) - a_j\big) > 0$$

We now have:

$$v\big((x - \sum_{i<j} a_i \cdot p^{n_i}) - a_j \cdot p^{n_j}\big) > 0 + v(p^{n_j}) = n_j$$

and $n_j > n_{j-1}$, which finishes the construction.
Since the sequence $(n_j)_{j\in\mathbb{N}}$ is increasing, the sequence

$$\Big(\sum_{i<j} a_i \cdot p^{n_i}\Big)_{j\in\mathbb{N}}$$

is Cauchy. Moreover:

$$\lim_{j\to\infty} \big(\sum_{i<j} a_i \cdot p^{n_i}\big) = x$$

Hence $x \in \mathbb{Q}_p$, as required. □

Definition. A valued field (K,v) is **henselian** if for all $f \in V_K[X]$ and $a \in V_K$ such that $v(f(a)) > 0$ and $v(f'(a)) = 0$, there exists $x \in V_K$ such that $f(x) = 0$ and $v(x-a) > 0$.

By Hensel's lemma, $\mathbb{Q}_p$ is henselian.

Theorem 5. *(K,v) henselian $\Leftrightarrow$ the valuation of K extends uniquely to any algebraic extension.*

Remark. Consider (K,v) henselian. Let $(K,v) \subseteq (L,v)$ be a finite extension, then:

$valL/valK$ is torsion. Moreover, $[valL : valK] \leqslant [L : K]$.

$[resL : resK]\cdot[valL : valK] \leqslant [L : K]$. Hence, $[resL : resK] \leqslant [L : K]$.

[1] In the classical setting, where one extends the valuation by $v(0) := \infty$, this amounts to say $v(x-a_0) < \infty$.

Lemma. *Suppose $L/\mathbb{Q}_p$ is a finite extension, then:*

$$[resL : res\mathbb{Q}_p] \cdot [valL : val\mathbb{Q}_p] = [L : \mathbb{Q}_p]$$

Definition. Fix prime p.
(K, v) is **p-adically closed (pCF)**, if

1) K has characteristic 0.

2) $valK$ is a $\mathbb{Z}$-group (that is, $valK$ is elementary equivalent to $(\mathbb{Z}, +, <)$).

3) $v(p)$ is the least positive element in $valK$.

4) $resK = \mathbb{F}_p$.

5) (K, v) is henselian.

Remark. $\mathbb{Q}_p$ is p-adically closed. Hence, pCF is consistent.

L_v is the language of valued fields (K, V).

Theorem 6 (Macintyre, 1976)**.** *pCF admits QE in the language L_v plus a unary predicate P_n for each $n \geqslant 2$, where $P_n(x) \leftrightarrow \exists y(y^n = x)$.*

3 Sketch of a proof of Theorem 6

We use the following criterion:

Fact. A theory T has QE $\Leftrightarrow \forall A \leqslant \mathcal{M} \models T$, $T \cup \Delta(A)$ is complete.

In order to futfill this criterion, it is enough to show:

1. pCF is model complete.
2. pCF has unique weakly prime model extensions (where $\mathcal{M} \geqslant A$ is weakly prime over A if $\forall M \models$ pCF, $A \hookrightarrow M \Rightarrow \mathcal{M} \underset{A}{\hookrightarrow} M$)

Existence and uniqueness of p-adic closures

Proposition 1. *Let $E \subseteq K \models pCF$ be relatively agebraicaly closed in K, then $E \models$* pCF

Proof. Points 1), 3), 4) are clear. For point 5), use theorem 5: v has a unique extension to any algebraic extension of E for if it wouldn't, it would have more than one extension to an algebraic extension of K. Of course, as E is relatively algebraicaly closed in K, $P_n(E) = E^n$.
So the only thing to check is point 2). A $\mathbb{Z}$-group is either $\mathbb{Z}$ or a dense linearly ordered set of copies of $\mathbb{Z}$. As $v(p) \in valE$ remains the smallest positive element, there is a copy of $\mathbb{Z}$ around any element $v(x)$ of $valE$, namely $\{v(p^n \cdot x) \mid n \in \mathbb{Z}\}$. If $valK = \mathbb{Z}$, we are done since $val(E)$ is then

equal to $val(K)$. So suppose $val(K)$ is a dense linearly ordered set of copies of $\mathbb{Z}$. Suppose that $v(x)$ and $v(y)$ are elements of $valE$ which lie in different copies of $\mathbb{Z}$. Then for $i = 0$ or $i = 1$, $v(x) + v(y) + i$ is divisible by 2. For this i, let $z \in K$ be such that $v(z) = \frac{v(x)+v(y)+i}{2}$, i.e. $z^2 \cdot V^\times = x \cdot y \cdot p^i \cdot V^\times$. Now, z must lie in a third copy of $\mathbb{Z}$ contained between the first two. Hence, $val(E)$ is a dense linearly ordered set of copies of $\mathbb{Z}$, since $z = \sqrt{x \cdot y \cdot p^i} \in E$. $\square$

This proposition grants us the existence of p-adic closures. Write k^{p-adic} for the relative algebraic closure of k in K.
For the uniqueness, note that for (L, v) a finite extension of $(K, v) \models$ pCF, we have:

$$[resL : resK] \cdot [valL : valK] = [L : K]$$

In particular, if $(L, v) \models$ pCF then we must have $[L : K] = 1$, whence $L = K$. This proves the uniqueness of p-adic closure as a weakly prime extension.

Model completeness of pCF Let $K \subseteq K'$ be models of pCF. Let $f_i, g, g_i, h_i \in K[\bar{X}]$. Suppose that $\bar{a} \in K'$ realises

$$\phi(\bar{x}) := \bigwedge_i f_i(\bar{x}) = 0 \wedge g(\bar{x}) \neq 0 \bigwedge_i P_{n(i)}(g_i(\bar{x})h_i(\bar{x})^{-1}) \bigwedge_j V(g_j(\bar{x})h_j(\bar{x})^{-1})$$

Up to considering $K \subseteq K(x_1)^{p-adic} \subseteq K(x_1, x_2)^{p-adic} \subseteq \dots$ for a transcendance basis $(x_1, x_2, \dots)$ of K' over K, we can wlog assume $trdegK'/K = 1$. We will distinguish two cases.

Case 1): $valK = valK'$
Let $b \in K' \setminus K$ such that $K' = K(b)^{p-adic}$. The fact that $b \in K'$ is transcendantal over K together with the data $D_b := \{(k_0, k_1) \in K \mid v(b-k_0) = v(k_1)\}$ determines $K(b)$ as a valued field up to (K, v)-isomorphism in the language of pCF, hence:

$$\text{pCF} \cup \Delta(K, v) \cup \{t \neq k \mid k \in K\} \cup \{v(t-k_0) = v(k_1) \mid (k_0, k_1) \in D_b\} \models \exists \bar{x} \phi(\bar{x})$$

Thus, a finite fragment of this already implies $\exists \bar{x} \phi(\bar{x})$. As this finite fragment must be realised in K, we have $K \models \exists \bar{x} \phi(\bar{x})$.

Case 2): $valK \subsetneq valK'$
Let $b \in K'$ be such that $v(b) \notin valK$. Then we have $K(b)^{p-adic} = K'$. The data $D := \{(k_0, k_1) \in K \mid v(k_0) < v(b) < v(k_1)\}$ together with the $D_n := \{e \in \mathbb{N} \mid K' \models P_n(e \cdot b)\}$ determine $K(b)$ up to K-isomorphism in the language of pCF. We can now conclude as in case 1).

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2 Notions of minimality in valued fields

These notes are intended to be complementary to the chapter 2 of the talk([SLD]) given by Dugald Macpherson. The main references are [PLY], [SLD] and [MH]. These notes don't extend the whole subject discussed by Macpherson. More precisely I will prove of that ACVF is C-minimal and I will give more detailed information about the proof of that a C-minimal field is algebraically closed. For definitions please refer to [SLD]. For C-minimality, to prove the first theorem, I'm based rather on the definition of Macpherson given during the talk while the original definition is:

Definition. A C-relation on a structure is a ternary relation $C(x, y, z)$ such that

i. $\forall x \forall y \forall z \;\; C(x, y, z) \rightarrow C(x, z, y)$

ii. $\forall x \forall y \forall z \;\; C(x, y, z) \rightarrow \neg C(y, x, z)$

iii. $\forall x \forall y \forall z \forall w \;\; C(x, y, z) \rightarrow (C(w, z, y) \vee C(x, w, z))$

iv. $\forall x \forall y \;\; x \neq y \rightarrow \exists z \neq y \; C(x, y, z)$

A complete theory T is C-minimal if for every $M \models T$, every M-definable subset of M is definable by a quantifier free formula involving only the C-relation and equality. A valued field has a natural C-structure by interpreting $C(x, y, z)$ by: $v(x - y) < v(y - z)$.

For the second theorem,(*If K is a C-minimal valued field then it is algebraically closed*), we follow the steps presented in [MH], where the original notion of C-minimalty is used. In fact it is possible to omit the hypothesis that K is originally a valued field. One can prove that a C-field, that is, an L-structure where L contains field language and the C-relation symbol, which is a field verifying above definition, is a valued field satisfying $C(x, y, z) \longleftrightarrow v(x - y) < v(z - y)$(see [MH] proposition 5.1).

In the end I'll give the proof of the fact that C-minimal theories has NIP(non independence property). Note that by a theorem from a Shelah, a complete theory T is unstable if, and only if, it has independence property or strict order property. I'll first give a proposition which characterize NIP in terms of indiscernible sequences. For a proof please refer to(for example) [Ad].

Notations

If K is a valued field, we'll note $\mathcal{O}_{\mathcal{K}}$, the valuation ring of K, M_K its maximal ideal, and k its residue field. If the context is clear we will omit index letters. For ultrametric spaces(see definition in [SLD]), the predicate introduced by Macpherson noted by val, here be noted simply by v. Finally a boolean combination will be understood always as finite boolean combination.

Theorem 7. *ACVF is C-minimal.*

Proof. Let $K \models$ ACVF. We have to show that every definable subset(with parameters) of K is a boolean combination of balls. Recall that ACVF admits quantifier elimination in the language of rings augmented by a predicate div(cf. [ChC] , chapter 2), interpreted by: $x \; div \; y$ if and only if $v(x) \leq v(y)$. So, it is sufficient to show our assertion for positive atomic formulas. Any positive atomic formula in single variable x with parameters in K, is of the one of the following forms:

$$p(x) = q(x) \quad ; \quad p(x) \; div \; q(x),$$

where $p, q \in K[X]$. The first one defines either the whole field, or a finite set of points(closed balls). We can split the second into two formulas:

$$v(p(x)) < v(q(x)) \quad \text{or} \quad v(p(x)) = v(q(x)).$$

As K is algebraically closed p and q split into linear factors. Write $p = dp'$ and $q = eq'$ with p', q' monic and $e, d \in K$. Thus, above formulas are equivalent respectively to: $v(d) + \sum_{i \in I} v(x - \alpha_i) < v(e) + \sum_{j \in J} v(x - \beta_j)$ and $v(d) + \sum_{i \in I} v(x - \alpha_i) = v(e) + \sum_{j \in J} v(x - \beta_j)$, where α_i and β_j are roots of p and q respectively. Let $L = I \cup J$ and $\gamma_l = \alpha_l$ if $l \in I$, $\gamma_l = \beta_l$ if $l \in J$. So, we reduce our formulas to:

$$\sum_{l \in L} \epsilon_l v(x - \gamma_l) \underset{<}{=} \delta,$$

where, $\delta = v(e) - v(d)$ and ϵ_l is '$-$' if $l \in J$. Let's show our assertion for the formula

$$\sum_{l \in L} \epsilon_l v(x - \gamma_l) \; < \delta. \tag{1}$$

The other case can be easily deduced from this.

Let $x \in K$. Define the equivalence relation E_1 in L by iEj, if and only if $v(x-\gamma_i) = v(x-\gamma_j)$. Let M_x be the maximal class with respect to valuation. That means $M_x = \{l \in L \mid \forall l' \ v(x-\gamma_l) \geq v(x-\gamma_{l'})\}$. Then, define the equivalence relation E_2 on K by xE_2y if, and only if $M_x = M_y$. So A_{M_x}, the class of x under E_2 is of the form

$$\left\{ y \in K \mid \bigwedge_{i,j \in M_x} (v(y-\gamma_i) = v(y-\gamma_j)) \wedge \bigwedge_{i \in M_x} \bigwedge_{j \notin M_x} v(y-\gamma_i) > v(y-\gamma_j) \right\}.$$

Let $i \in M_x$. Note that for all $j \notin M_x$ and $y \in A_{M_x}$ $\delta_j := v(\gamma_i - \gamma_j) = v(\gamma_i - y + y - \gamma_j) = v(y-\gamma_j)$. On the other hand assume following claim.
Claim. Let $M \subset L$. A subset defined by a formula of the form $\bigwedge_{i,j \in M} v(y-\gamma_i) = v(y-\gamma_j)$ is boolean combination of balls. Thus, the set A_{M_x} is boolean combination of balls.
Let $y \in A_{M_x}$. In the formula 1, for $j \notin M_x$, we can replace any occurrence of $v(x-\gamma_j)$ by some δ_j. As for all $i, j \in M_x$, $v(y-\gamma_i) = v(y-\gamma_j)$, the formula 1 is satisfied by y if and only if $kv(y-\gamma_{i_0}) > \delta - \sum_j \epsilon\delta_j$, for some integer k. As the value group is divisible the latter formula defines a ball.
Then, remark that under E_2 there is only a finite number of equivalence class $A_0 \dots A_n$. Each class is a boolean combination of balls. Furthermore by the above argument on each class there is formula $\phi_i(x)$ defining a ball. Hence the formula 1 is equivalent to the disjunction of formulas, $x \in A_i \wedge \phi_i(x)$.
Proof of the claim. By induction on the cardinality of the index set M, it is sufficient to prove for $|M| = 2$. In fact, let $i_0, j_0 \in M$, $\bigwedge_{i,j \in M} v(x-\gamma_i) = v(x-\gamma_j)$ if and only if,

$$\left(\bigwedge_{i,j \in M \setminus \{i_0\}} v(x-\gamma_i) = v(x-\gamma_j) \right) \wedge v(x-\gamma_{i_0)} = v(x-\gamma_{j_0}).$$

So, let $\alpha, \beta \in K$ and consider the formula $\phi(x) : v(x-\alpha) = v(x-\beta)$. By ultrametric triangular inequality $v(x-\alpha) > v(\alpha-\beta)$ implies that $v(x-\beta) = v(\alpha-\beta)$. This means, for all $x \in \phi(K)$, $v(x-\alpha) \leq v(\alpha-\beta)$. Thus, $\phi(K) = \{x \in K \mid v(x-\alpha) < v(\alpha-\beta) \vee v(x-\alpha) = v(\alpha-\beta) = v(x-\beta)\}$, which is a boolean combination of balls. □

Theorem 8. *If K is a C-minimal valued field , then K is algebraically closed.*

Proof. We will give some indications on the proof of theorem C, presented in [MH].

Step 1. We prove by C-minimalty that the value group is o-minimal, hence divisible.

Step 2. We prove that if char $K = 0$ then, $x^n = \alpha$ has solution for all n and $\alpha \in K$, by showing $(K^*)^n = K^*$. By divisibility of value group, we get $v((K^*)^n) = v(K^*)$. Remark that $(K^*)^n$ is a definable subgroup of K^*. By C-minimalty, this is a finite union of swiss cheeses which contains for all γ an element of valuation γ . Then we deduce that for all but finitely many γ, $(K^*)^n$ contains all open balls of radius γ(not containing 0). By translations we finish to show that $(K^*)^n = K^*$.

If char $K = p \neq 0$, with approximately the same argument above we prove that $\{x^p - x \mid x \in K\} = K$.

Step 3. We prove that the residue field of K, k, is algebraically closed. It turns out by C-minimality that k is strongly minimal. By a result of Macintyre (see [Mac]), a strongly minimal infinite field is algebraically closed. Thus, it is sufficient to show that k is infinite. Note that this is immediate by definition given by Macpherson during his talk, as indicated in [SLD]. To prove suppose k is finite. We have 3 cases:$|k| = q > 2$; $|k| = 2$ et char$K = 2$; $|k| = 2$; char$K = 0$. For the first case, pick an $\alpha \in \mathcal{O}$ which is different from $0, 1$ modulo M. Let x such that $x^{q-1} = \alpha$(possible by second step). A such x is in $\mathcal{O}$. We get a contradiction because $\bar{x}^{q-1} = \bar{\alpha} \neq \bar{1}$. Other cases are similar, for the second consider $x \in K$ such that $x^2 - x = 1$ and for 3rd, by using the fact that every element has a square root, show there is an x sucht that $x^2 + x + 1 = 0$.

As a consequence of what we did until now, we obtain that every algebraic extension of K is immediate, that is, it does not extend value group or residue fields.(cf [Kap], for definitions).

Step 4. We show that K is dense in its henselisation (K^h, v_h). Remark that for our purpose we can suppose that K is countable. In this case, the henselisation K^h of K is the union of algebraic extensions, $(K_i)_{i\in\omega}$, ordered by inclusion, such that, every extension $K_{i+1} : K_i$ is obtained by adjoining a *simple* root α_i of a polynomial $f_i = X^n + a_{i,1}X^{n-1} + \cdots + a_{i,n} \in \mathcal{O}_{K_i}[X]$ verifying $a_{i,1} \notin M_{K_i}$ and $a_{i,j} \in M_{K_i}$ for $2 \leq j \leq n$, with $\alpha_i - a_{i,1} \in M_{K_{i+1}}$. The henselisation of K is a immediate extension of K.

Suppose K is not dense in K^h, then there is a least i such that K is not dense in K_i. Let $\alpha \in K_i$, a root of $f := f_{i-1}$ as above.(i.e. $K_i = K_{i-1}(\alpha)$). In this case, for all sequence $(\alpha_\lambda)_{\lambda\in\kappa}$ indexed by an ordinal κ, we have

$$\exists c \in \Gamma\ \forall\lambda\ \exists\mu\ \ (\mu > \lambda \wedge v(\alpha_\mu - \alpha) \leq c).(*)$$

The idea is to contradict $(*)$, by finding a subset S of K_{i-1} that approximate α. That is for all $\beta \in S$ to get $v(\beta - \alpha) \geq \gamma_\beta$, with $(\gamma_\beta)_{\beta\in S}$ cofinal in Γ and to use the density of K in K_{i-1}.

Namely the subset S is the set of β of valuation 0 such that $\overline{\beta}$ is a simple root of $\overline{f}$. The main property of S is the fact that it is closed under *Newton approximation*. This means, $\beta \in S \Rightarrow \beta - f(\beta)/f'(\beta) \in S$. By using this property, we show that Δ, the convex hull of $v(f(S)) \cup -v(f(S))$, is a subgroup of Γ. By density of K in K_{i-1} and by continuity of f, we show that every open interval $(0, v(f(\beta)))$ in Γ can be defined by $\beta \in K$(thus $K \cap S$). This implies that Δ is definable in K. Hence by o-minimality, $\Delta = \Gamma$. Then we prove that $v(f(\beta))$ fit for the sequence γ_β by using again the main property of S.

Step 5. We consider the unique maximal immediate extension K^m of K, and prove that it is algebraically closed. Thus we get, K is dense in its algebraic closure. The uniqueness of K^m is always verified for fields of characteristic 0, for the characteristic p it follows easily from following criteria equivalent to Kaplansky's Hypothesis A(see [FVK]): The value group is p-divisible and the residue field is p-closed. Where p-closed means that every polynomial of the form $a_n X^{p^n} + a_{n-1} X^{p^{n-1}} + \cdots + a_1 X - a_0$ has a root.

The proof is based on following lemma in [MMV](lemma 7),

Lemma. *If K is a perfect henselian-valued field satisfaying following assertions then K is algebraically closed.*

a. The residue field k, of K is algebraically closed.

b. The value group of K is divisible.

c. If char(K)=0 and char(k)=p then K^* is p-divisible.

d. If char(K)=p, for all $\alpha \in K$ the polynomial $X^p - X - \alpha$ factorise completely.

As *a.* and *b.* follows from previous steps, we need to verify only *c.* and *d.* For this we essentially use the density of K in K^m and some aspects of cellular decomposition. Note that the use of cellular decomposition is questioned by the authors of the paper([MH], see remarks after the proof of theorem 5.9).

Step 6. We prove $K^{alg} = K$. Suppose this is not the case. As K is perfect by second step, K is not separably closed. Let α separable $\in K^{alg} \setminus K$ and $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_n$, distinct conjugates of α.

Define following objects;

$g(Y, x_0, \ldots, x_{n-1})$: The monic polynomial $P(Y)$ of degre n with coefficient of Y^j for $j < n$ equal to x_j.
$G(Y, z_1, \ldots, z_n)$: The monic polynomial $Q(Y)$ of degre n with roots

$\gamma_j = \sum_{i=1}^{n} z_i \alpha_j^i$.
$G_i(\overline{z})$: Coefficient of Y^i, $i < n$, in $G(Y, z_1, \ldots, z_n)$.
$f : (K^{alg})^n \ni \overline{\delta} \mapsto \in (K^{alg})^n$.
Thus

$$g(Y, f(\overline{z})) = G(Y, z_1, \ldots, z_n).$$

$X = f(K^n \setminus \ K)$ and $\widetilde{X} = f((K^{alg})^n \setminus K^{alg})$. Here K(respectively K^{alg}) is identified with n-tuples with last $n-1$ coordinates are 0.

Remark that f is continuous, and if $x \in X$ and z is such that $f(z) = x$ then $g(Y, x) = G(Y, z)$ has no roots in K. We will find a contradiction due to following claims.

Claim 1: X doesn't have interior in K^n.
We use above remark and prove that $\{x \mid g(Y, x) \text{ has no roots}\}$ has no interior in X(see lemma 5.8 [MH]).
Claim 2: $\widetilde{X}$ contains a non-empty open subset of K^{alg}.

By claim 2, there exits open balls $B_1 \times \cdots \times B_n \subset \widetilde{X}$. By density of K in K^{alg}, $\prod(D_i := B_i \cap K) \subset K^n$ is open in K^n. By a cellular decomposition theorem (theorem 4.4 from [MH]), there exists open balls E_i of K, $D_i \subset E_i$ such that $U' := \prod E_i \subset \widetilde{X} \setminus X$. So, the closure of U' contains an open subset $U \subset \widetilde{X} \setminus X$. Then, $V := f^{-1}(U)$ is open, so by density there is $\overline{\delta} \in V \cap (K^n \setminus K)$. But $f(\overline{\delta}) \in X$ so $\notin U$. Contradiction.

□

2.1 NIP

The proof of following proposition can be found in [Ad].

Proposition 2. *A complete theory T has NIP, if and only if, for all $M \models T$, for all order-indiscernible sequences $(a_i)_{i\in\omega}$ over $\emptyset$, and for all M-definable subset $S \subset M$, the set $\{i \mid a_i \in S\}$ is either finite or cofinite.*

Theorem 9. *A complete C-minimal theory has NIP.*

Proof. Let M be a model of a such theory and $S \subset M$ a boolean combination of balls. The proof goes by induction on the complexity of formula defining S. So we can assume that S is a ball. Let say open for now. That is, $S = \{x \mid v(x, a) > \gamma\}$, with $a \in VF$ and $\gamma \in \Gamma$. Let $(a_i)_{i\in w}$ be an indiscernible sequence of M. There are 3 cases:
1. $v(a_0, a_1) < v(a_1, a_2) < \cdots < v(a_i, a_{i+1}) < \ldots$
2. $v(a_0, a_1) > v(a_1, a_2) > \cdots > v(a_i, a_{i+1}) > \ldots$
3. $v(a_0, a_1) = v(a_1, a_2) = \cdots = v(a_i, a_{i+1}) = \ldots$

In the first case, if there is some $a_i \in S$ then for all j such that $v(a_i, a_j) > \gamma$, a_j is in S by transitivity. But if there is such a j then for all $k > j$ $v(a_k, a_j) > \gamma$ so $a_k \in S$. The other cases are similar. $\square$

3 Imaginaries in ACVF

Imaginaries in valued fields have first been investigated by Holly and Mayintyre-Scowcroft.

Definition.

1. A complete theory T has elemination of imaginaries (EI) if for all $M \models T$, $n > 0$ and $\emptyset$-definable equivalence relations E on M^n, there is $m > 0$ and an $\emptyset$-definable function $f : M^n \to M^m$ such that for all $x, y \in M^n$
$$E(x, y) \iff f(x) = f(y).$$
2. An imaginary is an equivalence class of some $\emptyset$-definable equivalence relation on some M^n.

Given $M \models T$, you can consider the model M^{eq} (in language L^{eq}, with theory T^{eq}) you get by adding, for each E on M^n like above, a sort M^n/E and a function $\pi : M^n \to M^n/E$ identifying it.

Fact. $\mathrm{Th}(M^{eq}) = T^{eq}$ has elimination of imaginaries.

But while passing to M^{eq}, we loose control of definability; so one aims to get EI by adding only specific, well-understood sorts from M^{eq} to M.
The idea of EI is the following: Let $R \subseteq M^n \times M^t$ be a $\emptyset$-definable relation. This can be understood as a family of definable sets in M^t, with parameters in M^n; for $a \in M^n$, let $R_a := \{y \in M^t \mid R(a, y)\}$. Now define an $\emptyset$-definable equivalence relation E_R on M^n by

$$E_R xy \iff R_x = R_y.$$

The imaginary $x/E_R \in M^{eq}$ codes R_x, we write $x/E_R = \ulcorner R_x \urcorner$. Given EI, there is $m > 0$ and a $\emptyset$-definable $f : M^n \to M^m$ such that

$$f(x) = f(y) \iff E_r xy \iff R_x = R_y.$$

So now $f(x)$, which is a tuple from M, can be viewed as a code for R_x. Put $Z := \mathrm{Im}(f)$; you can replace R by $R' \subseteq Z \times M^t$, where

$$R(x, y) \iff R'(f(x), y).$$

R and R' define the same family of sets in M^t, but with respect to R', every set has a unique parameter.

Remark. EI means that for each imaginary e of the monster model $\mathcal{U}$ (i.e. $e \in \mathcal{U}^{eq}$)

$$e \in \mathrm{dcl}(\mathrm{dcl}(e) \cap \mathcal{U}).$$

A natural weakening of this is the so-called weak elimination of imaginaries (weak EI):

$$e \in \mathrm{dcl}(\mathrm{acl}(e) \cap \mathcal{U}).$$

(where dcl and acl are understood in the sense of M^{eq}).

Example.

1. The theory of infinite pure sets doesn't have EI, since it is not possible to code finite sets (with more than one element): Consider the equivalence relation $(x, y)E(x', y') \iff \{x, y\} = \{x', y'\}$. For some imaginary element $e = \{a, b\}$ always $\mathrm{dcl}(e) \cap \mathcal{U} = \emptyset$ (since a, b are not definable over e, and all other elements or tuples are not as well), so $e \notin \mathrm{dcl}(\mathrm{dcl}(e) \cap \mathcal{U}) = \mathrm{dcl}^{eq}(\emptyset) = \emptyset$. But (a, b) is algebraic over e, and e is definable over (a, b), so the theory does have weak EI. The same holds for vector spaces in the language of modules.

2. RCF (or any o-minimal expansion of it) has EI: For any $\emptyset$-definable equivalence relation E on M^n there is a $\emptyset$-definable function $f : M^n \to M^n$ picking out a representative for each E-class (this holds by definability of Skolem functions; for example, as a canonical representative of the interval bounded (a, b), you can choose the midpoint of the interval, which is definable because of the additive group structure).

3. ACF has EI: First, you can code finite sets $\{a_1, \ldots, a_n\}$ by the sequence of coefficients of $\prod(X - a_i)$; with a slight generalisation (using Chow coordinates) you can even code finite sets of tuples. Now you can either use QE and the existence of a unique smallest field of definition of an affine variety and argue algebraically, or prove first weak EI and then EI by purely model theoretic arguments (for this, see for example section 3.2 of [Ma]).

4. Any completion of ACFA (the theory of algebraically closed fields with a generic automorphism) has EI, see [CH].

Remark. ACVF does not have EI in the sort K (in the language $L_v = \{+, \times, -, 0, 1, |\}$: The residue field k is (as a definable subset of M^{eq}) stably embedded into K; it's an algebraically closed field and so strongly minimal. If ACVF had EI, there would be an definable subset X of K isomorphic to k; but no infinite definable subset of K is stable.
Also in the sorts (K, Γ, k) (in the appropriate language $L_{k,\Gamma}$), ACVF does not have EI (see [Ho])

Initially, there was the hope one gets EI by adding just sorts for open and closed balls. This is false; there are some more complicated sorts needed.

Definition. An n-lattice is a free rank n R-submodule of K^n (so an n-lattice will have the form $Rv_1 \oplus \ldots \oplus Rv_n$ for some K-linearly independent $v_1, \ldots, v_n \in K^n$).

Let Z_n be the set of ordered bases $(z_1, \ldots, z_n)$ of K^n, so $Z_n \subseteq K^{n^2}$. Define an ($\emptyset$-definable) equivalence relation E_{S_n} on Z_n:

$$(x_1, \ldots x_n) E_{S_n} (y_1, \ldots, y_n) \iff \sum Rx_i = \sum Ry_i.$$

Then the equicalence classes of E_{S_n} correspond to n-lattices. Let

$$\underline{S_n = \{ \text{ set of equivalence classes of } E_{S_n} \}};$$

this is then a set of codes for n-lattices. There is another interpretation of S_n as a coset space:
$\mathrm{GL}_n(K)$ acts transitively (by left multiplication) on the set of ordered bases of K^n, hence transitively on the set of n-lattices. One of these n-lattices is R^n with standart basis $e_1, \ldots, e_n$. The stabiliser of R^n is just $\mathrm{GL}_n(R)$. So by the orbit-stabiliser theorem, the space of n-lattices is naturally in bijection with the coset space $\mathrm{GL}_n(K)/\mathrm{GL}_n(R)$.
Finally, we can also identify S_n with $\mathrm{B}_n(K)/\mathrm{B}_n(R)$, where $\mathrm{GL}_n(K)$ is the group of invertible upper-triangular matrices.

Remark. S_1 is in bijection with Γ:

$$S_1 = \mathrm{GL}_1(K)/\mathrm{GL}_1(R) = K^*/R^* \cong \Gamma,$$

so S_1 has a natural group structure.

Note each 1-lattice has the form $\gamma R = \{x \in K \mid v(x) \geq \gamma\}$; so 1-lattices correpond to closed balls with center 0. For $n \geq 2$, $\mathrm{GL}_n(R) \not\trianglelefteq \mathrm{GL}_n(K)$, so there is no natural group structure on S_n.

Remark. We may also see infinite closed balls not containing 0 as elements of S_2:
Define E on $\{(x, y) \in K^2 \mid x \neq y\}$ by

$$(x, y)E(x', y') \iff B_{\geq v(x-y)}(x) = B_{\geq v(x'-y')}(x').$$

A closed ball not containing 0, say $B_{\geq\gamma}(a)$ where $v(a) < \gamma$ is a torsor (this is, a one-dimensional affine space) for γR:

$$B_{\geq\gamma}(a) = B_{\geq\gamma}(0) + a.$$

We can identify $B = B_{\geq\gamma}(a)$ with an element of S_2: $\{1\} \times B$ generates an R-submodule L. Now

- $B = L \cap (\{1\} \times K)$
- L is a 2-lattice.

If L is an n-lattice, then $\mathcal{M}L := \{ax \mid a \in \mathcal{M}, x \in L\}$ is an R-submodule of L. The quotient

$$\underline{\mathrm{red}(L) := L/\mathcal{M}L}$$

has the structure of an n-dimensional vector space over $R/\mathcal{M} = k$. In particular, if $L = \gamma R = aR$ (where $a \in K$ with $v(a) = \gamma$)) is a 1-lattice, then

$$\begin{aligned}\mathrm{red}(L) &= \gamma R/\gamma\mathcal{M} \\ &= \{\gamma\mathcal{M} + a \mid v(a) \geq \gamma\} \\ &= \{B_{>\gamma}(a) \mid v(a) \geq \gamma\}.\end{aligned}$$

Here $\mathrm{red}(L)$ is the set of open subballs of γR of radius γ. It is a torsor of k, so it is strongly minimal.

The set $\bigcup_{\ulcorner L \urcorner \in S_n} \mathrm{red}(L)$ is a for every n uniformly definable family of subsets of K^n. Define

$$\underline{T_n := \{\text{ codes for members of this family}\}}.$$

You can see T_n as a quotient, by E_{T_n}, of

$$X_n := \{(x, y_1, \ldots, y_n) \in K^{n(n+1)} \mid y_1, \ldots, y_n \text{ lin. indep.}, x \in Ry_1 \oplus \ldots \oplus Ry_n\}$$

where

$$\begin{aligned}(x, y_1, \ldots, y_n)E_{T_n}(x', y_1', \ldots, y_n') \iff & Ry_1 \oplus \ldots \oplus Ry_n = Ry_1' \oplus \ldots \oplus Ry_n' \\ & \text{and } x + \mathcal{M}L = x' + \mathcal{M}L.\end{aligned}$$

We have an $\emptyset$-definable surjection $\pi_n = T_n \to S_n$, $\pi_n((x, y_1, \ldots, y_n)/E_{T_n}) = (y_1, \ldots, y_n)/E_{S_n}$. The fibres of π_n form a set of codes for elements of some $\mathrm{red}(L)$.
It is $T_1 = \{\ulcorner B_{>v(a)}(a) \urcorner \mid a \in K^*\}$, and

$$\{B_{>v(a)}(a) \mid a \in K^*\} = \{a(1 + \mathcal{M}) \mid a \in K^*\} = K^*/1 + \mathcal{M} = RV.$$

So, like S_1, also T_1 is a group.
We can view T_n as a collection of coset spaces: For $m = 1, \ldots, n$, let

$$\begin{aligned}B_{n,m}(k) = \{A \in B_n(k) \mid & \text{the m-th column of } A \text{ has a 1} \\ & \text{as m-th entry, other entries } 0\},\end{aligned}$$

and

$$B_{n,m}(R) = \{A \in B_n(R) \mid, A \text{ reduces mod } \mathcal{M} \text{ to } B_{n,m}(k)\}.$$

Put $B_{n,0}(R) = B_n(R)$. Then each $B_{n,m}(R)$ is a group, and one can identify T_n with the union of cosets $\bigcup_{m=0}^{n} B_n(K)/B_{n,m}(R)$.

Definition. Let $L_{\mathcal{G}}$ be the multisorted language with sorts $K, k, \Gamma, (S_n)_{n\geq 1}$, $(T_n)_{n\geq 1}$; we refer to these sorts as to the sorts $\mathcal{G}$ (for *geometric* sorts). We use an appropriate language $L_{\mathcal{G}}$.

As we have seen before, the sorts k and Γ are redundant (since $k = \text{red}(R) \subset T_1$ and $\Gamma = S_1$). ACVF has QE in $L_{\mathcal{G}}$ with sorts $\mathcal{G}$ (see [HHM1], Theorem 3.1.2).

Theorem 10 (Haskell, Hrushovski, Macpherson)**.** *ACVF has EI in the sorts of $L_{\mathcal{G}}$.*

Remark.

1. Essentially, we need this level of complexity of sorts.
2. For every imaginary e, there is some R-submodule of some K^n with code interdefinable with e.
3. (Mellor, see [Me]) RCVF has EI in the same sorts.
4. (Hrushovski-Martin, see [HM]) $\text{Th}(\mathbb{Q}_p)$ has EI in the sorts $K, (S_n)_{n\geq 1}$.

<u>There are several Applications of EI for $\mathbb{Q}_p$:</u>
Let G be a finitely generated nilpotent group, let R_n be the set of irreducible complex n-dimensional characters. Define an equivalence relation on $R_n(G)$ via $\sigma_1 \sim \sigma_2 \iff$ there is a linear character χ with $\sigma_1 = \chi\sigma_2$.

Fact (Lubotzky-Magid, [LM])**.** $a_n(G) := \mid R_n(G)/\sim \mid$ is finite

Now define $\zeta_{G,p}(t) = \sum_{n=0}^{\infty} a_{p^n} t^n$.

Theorem 11 (Hrushovski-Martin, [HM])**.** *Let G be a finitely generated nilpotent group. Then, for any prime p, the zeta function $\zeta_{G,p}(t)$ is a rational function over $\mathbb{Q}$.*

Theorem 12 (Hrushovski-Martin, [HM])**.** *Let $(R_l)_{l\in\mathbb{N}^r}$ be a definable family of subsets of $\mathbb{Q}_p^N$. Let $E = (E_l)_{l\in\mathbb{N}^r}$ be a definable family of equivalence relations on $(R_l)_{l\in\mathbb{N}^r}$. Suppose that for each $l \in \mathbb{N}^r$, $\mid R_l/E_l \mid$ is finite. Put $a_l := \mid R_l/E_l \mid$.*
Then the power series $\sum_{l\in\mathbb{N}^r} a_l t^l \in \mathbb{Q}[[t_1, \ldots, t_r]]$ is $\mathbb{Q}$-rational.

There are some other applications for group-theoretic rationality results: Let G be a finitely generated torsion-free nilpotent group, p a prime, and b_n one of

- number of index p^n subgroups.
- number of conjugacy classes of index p^n subgroups.

Then $\sum b_n t^n$ is a rational function.
This was shown in 1988 by Grunewald, Segal, Smith; in [HM], it is reproven using EI for $\mathbb{Q}_p$.

4 Proof of EI in ACVF

The key issues with EI in ACVF are the following:

1. There is no straightforward independence theory (apart from that arising from algebraic closure, which is as for ACF).
2. Every element of a ball is a centre of it, so a ball does not have a canonical centre/parameter.

In the following, there are some aspects of the proof of EI described which have use beyond the proof itself (e.g. metastable fields, and work from Hrushovski- Kazhdan on integration in valued fields). We often work over a parameter set C; we'll assume $C = \mathrm{acl}^{eq}(C)$ for simplicity (it is only needed for some of the statements).

4.1 Description of 1-types

We describe 1-types in the sort K over the parameter set C (the arguments also apply to 'unary types' of the sorts S_n, T_n).
Let $a \in K$; define

$$\mathcal{B}_C(a) := \{B \mid B \text{ a } C\text{-definable ball, } a \in B\}$$

The balls in $\mathcal{B}_C(a)$ form a chain under inclusion (as for any balls B_1, B_2, $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$ or $B_1 \cap B_2 = \emptyset$). Define

$$\mathrm{Loc}_C(a) := \bigcap \{B \mid B \in \mathcal{B}_C(a)\}$$

(viewed as a subset of $\mathcal{U} \cap K$).

Remark. If $\mathcal{B}_C(a)$ has a smallest ball, $\mathrm{Loc}_C(a)$ equals this ball. Otherwise, $\mathrm{Loc}_C(a)$ is an ∞-definable set in K (over C).

- $b \in K$ is <u>generic</u> in $\mathrm{Loc}_C(a)$ if $\mathrm{Loc}_C(b) = \mathrm{Loc}_C(a)$, i.e. a and b lie in the same C-definable balls.
- If $C \subseteq D$, a is <u>generic in $\mathrm{Loc}_C(a)$ over D</u> if $\mathrm{Loc}_C(a) = \mathrm{Loc}_D(a)$; then we write $a \mathop{\smile\!\!\!\!\!\!\;|}^{g}_{C} D$.

Lemma. *Suppose $C \subseteq D$. Then*

1. *If B is a C-definable ball, or the intersection of a chain of C-definable balls, then $B = Loc_C(a)$ for some $a \in B$.*

2. *If b is generic in $Loc_C(a)$, then $a \equiv_C b$ (so $Loc_C(a)$ has a unique generic type realised by a).*

3. *$tp(a/C)$ has a unique generic extension over D.*

4. *If $a \equiv_C a'$, $a \mathop{\smile\!\!\!\!\!\!\vert}^{g}_{C} D$, $a' \mathop{\smile\!\!\!\!\!\!\vert}^{g}_{C} D$, then $a \equiv_D a'$ (stationarity).*

5. *If B is a ball, the generic type of B over $\mathcal{U}$ is definable over $\ulcorner B \urcorner$*

Remark: All this holds in any C-minimal theory in Hrushovski-Kazhdan sense.
Proof:

1. As B is not the union of finitely many proper sub-balls, one can use compactness to choose $a \in B$ outside any C-definable proper sub-balls of B.

2. Since definable subsets of K are boolean combinations of C-definable balls, any 1-type over C is determined by C-definable balls.

3. As for 1)

4. As for 2)

5. Suppose $B = B_{\geq\gamma}(a)$ is a closed ball of radius γ. The set of open sub-balls of B of radius γ (which is just red(B)) is strongly minimal, so for any formula $\phi(x, y)$ (with x a single field variable), there is n_ϕ such that if $\phi(x, a) \cap B$ lies in the union of finitely many elements of red(B), it is in a union of $\leq n_\phi$ elements of red(B); otherwise, $\phi(x, a)$ contains the union of all but $\leq n_\phi$ elements of red(B). So if p is the generic type of B,

$$\phi(x, a) \in p \iff \phi(x, a) \text{ meets more than } n_\phi \text{ elements of red}(B).$$

4.2 The stable part of ACVF

We work over a parameter set C.

Definition. A C-definable set D is stable if it is stably embedded, and is stable as a structure with C-definable relations.

There are various equivalent conditions:

- D is stable

- For any formula $\phi(x_1, \dots x_n, y)$ which implies $\bigwedge_{i=1}^n D(x_i)$, $\phi = \phi(\bar{x}, y)$ is stable.
- If $\lambda > |T| + |C|$, $\lambda = \lambda^{\aleph_0}$ and $B \supseteq C$ with $|B| = \lambda$, there are $\leq \lambda$ 1-types over B which are realised in D.

Definition. If D_1, D_2 are definable sets in $\mathcal{U}$, then D_2 is D_1-internal if there is finite F such that $D_2 \subseteq \mathrm{dcl}(D_1 \cup F)$
(so for some n, there is a definable surjection $D_1^n \to D_2$).

If $s \in S_n$, then red(s) means red(L) where L is the n-lattice coded by s. So red(s) is an n-dimensional k-vector space.

Lemma. *If $s \in S_n$, then red(s) is k-internal and stable (viewed as an $\ulcorner s \urcorner$-definable set).*

Proof: It is $\mathrm{red}(s) \subseteq \mathrm{dcl}(k \cup \{(\text{finite) basis for red}(s)\})$; as k is stable, so is red(s) (it has Morley rank n).

Definition. $VS_{k,C}$ is the many-sorted structure with a sort red(s) for each $s \in S_n \cap \mathrm{dcl}(C)$ (i.e. a sort for each C-definable lattice).

Equip $VS_{k,C}$ with all C-definable relations on products of sorts. Then $VS_{k,C}$ is a many-sorted stable structure. Any finite union of sorts has finite Morley rank.

Theorem 13. *(ACVF) Let D be a C-definable subset of $\mathcal{U}^{eq}$. The following are equivalent:*

1. *D is stable.*
2. *D is k-internal.*
3. *There is no definable surjection from D to an infinite subset of Γ.*
4. *D is k-analysable.*
5. *$D \subset dcl(C \cup VS_{k,C})$*

One point in the proof is the following: No infinite definable subset D of K satisfies the conditions: Indeed, if $D \subset K$ is infinite and definable, it contains a ball, so we may suppose it is a ball. Pick $d \in D$. Let $D = B_{>\gamma}(d)$; define $x \mapsto v(x - d)$. The range contains $\Gamma^{>\gamma}$, so 3 fails.

Theorem 14. *$VS_{k,C}$ has elimination of imaginaries.*

Idea of proof:

1. $VS_{k,C}$ is closed under duals, tensor products and exterior powers.

2. We may easily reduce to coding a definable subset of a k-vector space V =red(s). Now, via identification $V \leftrightarrow k^n$ (over a basis), we have a notion of 'Zariski closed' (which is independent of basis). Via QE in ACF, any definable set is a boolean combination of Zariski closed sets. Identify $k[X_1, \ldots X_n]$ with $S(V) := k \oplus V^* \oplus \sum_{i \geq 2} \mathrm{Sym}^i(V^*)$. Every Zariski-closed set is determined by the ideal vanishing in $K[X_1, \ldots, X_n]$, hence by a subspace U of some

$$S^n(V) := k \oplus V^* \oplus \sum_{i \geq 2}^{n} \mathrm{Sym}^i(V^*).$$

Pull U back to a subspace U' of

$$T^n(V) := k \oplus V^* \oplus \sum_{i \geq 2} \otimes^i(V^*).$$

Reduce to coding a subspace of some red(s). Using exterior powers, reduce to coding a 1-space in some red(s).

3. Use the following general Lemma about EI:

Lemma. *Let $\mathcal{U}$ be a sufficiently saturated multi-sorted structure, with a sort D such that $\mathcal{U} \subset (\mathcal{U} \cap D)^{eq}$. Suppose that for each sort S of $\mathcal{U}$, every 1-variable partial function $f : D \to S$ is coded in $\mathcal{U}$. Then Th($\mathcal{U}$) has EI.*

Proof: We show by induction on n that each definable $R \subseteq D^n$ is coded.
For $n = 1$, any $R \subset D$ is coded by id_R. For the inductive step, we have $R \subset D^{n+1}$ and $\pi D^{n+1} \to D$ (projection on the first coordinate). Put $Y = \pi(R)$.
For each $y \in Y$, the fibre $R_y \subseteq D^n$, so by induction it is coded by some tuple $h(y)$. By compactness, h is definable, and we may write Y as a disjoint union $Y = Y_1 \dot{\cup} \ldots \dot{\cup} Y_t$ such that $h|_{Y_i}$ is a map to a specific product of sorts . Now by assumption each $h|_{Y_i}$ is coded in S, and $(\ulcorner h|_{Y_1} \urcorner, \ldots, \ulcorner h|_{Y_t} \urcorner)$ codes R.

4. <u>The proof of EI:</u> Some other ingredients are:

   (i) Definable R-submodules of K^n are coded in $\mathcal{G}$.

   (ii) Definable functions $\Gamma \to \mathcal{G}$ (and their germs) are coded in $\mathcal{G}$.

   (iii) Finite sets of tuples from $\mathcal{G}$ are coded in $\mathcal{G}$.

Proposition 3. *Let B be a C-definable ball (or an intersection of a chain of C-definable balls). Let f be a definable partial function $K \to \mathcal{G}$ such that dom(f) $\supseteq B$, and assume $C = acl(C\ulcorner f \urcorner) \cap \mathcal{G}$ (e.g. if dom(f) $= B$ and $C = acl(\ulcorner f \urcorner) \cap \mathcal{G}$).*
Then there is a C-definable function g with the same germ on B as f.

Idea of proof: First proove a related result for germs of functions on closed balls, showing the germ is coded by a code for a definable R-module (use (i) from above). Then approximate B from inside by closed balls, using (ii).

Completion of the proof of EI:

We code definable partial functions $f : K \to \mathcal{G}$. Show that $\ulcorner f \urcorner \in \mathrm{dcl}(\mathrm{dcl}_{\mathcal{G}}(\ulcorner f \urcorner))$. Put $C := \mathrm{acl}_{\mathcal{G}}(\ulcorner f \urcorner)$, and

$$\Sigma := \{D \subset \mathrm{dom}(f) \mid D, f|_D \text{ are } C\text{-definable}\}.$$

If $\bigcup \Sigma = \mathrm{dom}(f)$, we're done by compactness. So suppose $\bigcup \Sigma \subsetneq \mathrm{dom}(f)$, and find a complete type p/C of realisations of $\mathrm{dom}(f) \setminus \bigcup \Sigma$. Then p is the generic type of a C-definable ball B (or the intersection of a chain of C-definble balls).
There is an C-definable map g with the same germ on B as f. Put

$$X := \{x \in B \mid f(x) = g(x)\};$$

then X contains all realisations of p. X is coded in $\mathcal{G}$ (as it is in 1 variable), and is $\ulcorner f \urcorner C$-definable, so is C-definable.
So C contains all realisations of p (as $p \in S(C)$). As g is C-definable, so is $f|_X$, so $X \in \Sigma$. This is a contradiction, as p is a type of elements of $\mathrm{dom}(f) \setminus \Sigma$.

How are modules used in the coding?
Suppose $t_1, \ldots t_m$ are open balls of radius γ, and there is $\delta < \gamma$ such that if $t_i \neq t_j$, $x \in t_i$, $y \in t_j$, then $v(x - y) = \delta$.
Put $F := \{t_1, \ldots, t_m\}$, let $T := t_1 \cup \ldots \cup t_n\}$. Define

$$J^F = \{f \in K[X] \mid \deg(f) \leq m, \forall x \in T : v(f(x)) > (m-1)\delta + \gamma\}.$$

Then J^F can be viewed as a definable R-submodule of K^{m+1}, so is coded in $\mathcal{G}$. Also $(\ulcorner J^F \urcorner, \gamma, \delta)$ is a code for F.
The point is: If $f \in K[X]$ is monic of degree m, then

$$f \in J^F \iff f \text{ has a root in each } t_i.$$

5 Stable Domination

The main idea is the following:
In ACVF, over a base C, we have a stable structure $VS_{k,C}$. What role does it play in understanding the types? We consider certain types which are governed by a 'trace' in $VS_{k,C}$. We will see some applications in section 6, e.g. to definable groups in ACVF (Hrushovski).
Most part of the material will be taken from Part I of [HHM].

Recall: A complete type p over M ($\mathcal{M}$ a model of a certain theory T) is *definable* over $C \subseteq M$, if for each formula $\phi(x,y)$, there is a formula $d_p\phi(x,y)$ over C, in the variable y, such that for any $a \in M$, $\phi(x,a) \in p$ iff $\models d_p\phi(x,a)$. That is, $d_p\phi(x,y)$ picks out the parameter a.
Now suppose $\mathcal{U} \models T$ is sufficiently saturated and homogeneous and $C \subseteq \mathcal{U}$ is small. Then $Aut(\mathcal{U}/C)$ acts on the space of types over $\mathcal{U}$: for $g \in Aut(\mathcal{U}/C)$, $\phi(x,a) \in gp \Leftrightarrow \phi(x, g^{-1}a) \in p$. We say $p \in S(U) is Aut(\mathcal{U}/C)$*-invariant* (or C*-invariant*, or *invariant*) if p is fixed by $Aut(\mathcal{U}/\mathcal{C})$.
$p \in S(U)$ is C-invariant $\Leftrightarrow$ p does not split over C, i.e. for any $a_1, a_2 \in \mathcal{U}$, if $a_1 \equiv_C a_2$, then

$$\phi(x,a_1) \in p \Longleftrightarrow \phi(x,a_2) \in p.$$

Example.

1. If $\mathcal{U} \models T_{DLO}$, then "$x > \mathcal{U}$" and "$x < \mathcal{U}$" determine the $Aut(\mathcal{U}/C)$ 1-types (and are the only such).
2. If $\mathcal{U}$ is the random graph, then the types saying "x is joined to $\mathcal{U}$" or "x is not joined to $\mathcal{U}$" are $Aut(\mathcal{U}/C)$-invariant (and are only such).
3. In $ACVF$, the generic type (over $\mathcal{U}$) of a C-definable ball, or of a chain of C-definable balls, is $Aut(\mathcal{U}/C)$-invariant .
4. If $\mathcal{U}$ is o-minimal (or just weakly o-minimal) then every type over C has an $Aut(\mathcal{U}/C)$-invariant extension (for 1-types, at most two such).

Fact.

1. Every C-definable type is $Aut(\mathcal{U}/C)$-invariant.
2. The converse is false: e.g., in $ACVF$, the generic type of a chain of ball with no least element (i.e. of the intersection of the balls) is C-invariant, not definable.
3. (Assume T is NIP) Every $Aut(\mathcal{U}/C)$-invariant type is *Borel definable* over C (cf. [HP]).
4. (Assume NIP) If every 1-type (in U) over every $C = acl^{eq}(C)$ has an $Aut(\mathcal{U}/C)$-invariant extension, then the same holds for all types (in U^{eq}) (cf. [HP]).
5. Every definable C-invariant type is C-definable.
6. Invariant types have Morley sequences, which are indiscernible.
7. (By facts 2 and 4) In $ACVF$, if $C = acl(C)$ then every type over C has a C-invariant extension.

5.1 The structure St_C

For convenience , assume T eliminates imaginaries and assume $C = acl(C)$ (though not needed in all assertions).

Definition. St_C is the many-sorted structure with a sort for each C-definable stable set, and with the C-definable relations on products of sorts endowed.

If $A \subseteq U, St_C(A) := dcl(CA) \cap St_C$, and likewise, if a is a tuple, possibly infinite, then $St_C(a) := dcl(Ca) \cap St_C$.

Remark. 1. The elements of C are singleton sorts of St_C, and lie in any $St_C(a)$.

2. In $ACVF$ St_C is 'essentially' $VS_{k,C}$. If $C \models ACVF$, then St_C is 'essentially' k.

Now we give several presentations for the notion of stably dominated type:

Stably dominated type 1. We say that $tp(a/C)$ is stably dominated if for any b we the following holds:

$$St_C(a) \downarrow_C St_C(b) \Longrightarrow tp(b/St_C(a)) \models tp(b/Ca).$$

Formally this means that for any b', if $b' \equiv_{St_C(a)} b$, then $b' \equiv_{Ca} b$.

Exercise. By automorphism argument (using that sorts in St_C are stably embedded) the statement above is equivalent to:

$$\text{for any } a',\ a' \equiv_{St_C(b)} a \Rightarrow a' \equiv_{Cb} a, \text{ i.e. } tp(a/St_C(b)) \models tp(a/Cb).$$

Stably dominated type 2. View $St_C(a)$ as an infinite tuple (with respect to some enumeration), so we have a map $f : a \mapsto St_C(a)$ on $tp(a/C)$. Let D be any definable set (in sort containing a) and $d := \ulcorner D \urcorner$. We say that a fibre X of f is D-generic if for any $a \in X$, $St_C(a) \downarrow_C St_C(d)$. Then $tp(a/C)$ is stably dominated iff, for each D, either D contains all D-generic fibres of f, or D is disjoint from all D-generic fibres of f (cf. compact domination).

Stably dominated type 3. $tp(a/C)$ is stably dominated iff, for any b with $St_C(a) \downarrow_C St_C(b)$, $tp(a/St_C(a))$ has a unique extension over $St_C(a)b$.

Remark. Without the assumption $C = acl(C)$, we just work with the first definition (the above equivalence use stationarity in St_C).

Example. ($ACVF$). Let p be the generic type over C of $R = B_{\geq 0}(1)$ (or of any closed ball). For simplicity, assume that C is a model, so St_C is essentially just k. If $a \models p$, then $res(a)$ is transcendental in k over $acl(C) \cap k$ (and essentially, $res(a)$ is $St_C(a)$). For any $B \supset C$, put $res(B) := dcl(B) \cap k$ (essentially this is $St_C(B)$). Then the folowing holds:

There is a unique complete type p' over B such that p' contains the formula '$x \in R$' and implies $res(x) \downarrow_C res(B)$.

This is true since otherwise there would be a formula $\phi(x)$ over B, which meet each open ball $\mathcal{M} + a$ (for a generic $a \in R$) in a proper non-empty set. This would contradict that $\phi(x)$ defines a boolean combination of balls. Thus, p is stably dominated. More generally, for any C-definable closed b, the generic type over C of b is stably dominated.

Theorem 15. *Let $C = acl(C)$.*

(i) If p is stably dominated, then p has a C-definable (so $Aut(\mathcal{U}/C)$-invariant) extension over U. In fact, p has a unique $Aut(\mathcal{U}/C)$-invariant extension.

(ii) If $p \in S(U)$ is $Aut(\mathcal{U}/C)$-invariant, and $C \subseteq B \subseteq U$, then $p|C$ stably dominated $\Rightarrow p|B$ stably dominated.

(iii) If $p, q \in S(U)$ are $Aut(\mathcal{U}/C)$-invariant, and $b \models q|C$, then $p|C$ stably dominated $\Rightarrow p|B$ stably dominated.

Question: We don't know if invariance of q is needed.

(iv) If $tp(a/C)$ and $tp(b/Ca)$ are stably dominated, so is $tp(ab/C)$.

We just give the proof of (i) modulo the following lemma:

Lemma. *Suppose C is small, and $r(x, y)$ is a type over U in possibly infinite tuples x, y. Let $q(x)$ be the restriction of r to the x variables. Suppose q is C-definable, and $r(x, y)$ is the unique extension over U of $r(x, y)|C \cup q(x)$.*
Then $r(x, y)$ is C-definable.

Proof. [of (i)] We want to prove definability of $p(y) \in S(U)$, with $p|C$ stably dominated.
Choose $a \models p$ with $St_C(a) \downarrow_C St_C(\mathcal{U})$. Let $r(x, y) = tp(St_C(a), a/U)$ and $q(x) = tp(St_C(a)/U)$. Since $q(x)$ is definable over C in the stable structure St_C, by the lemma r is C-definable, and hence so is p. □

5.2 Strong codes for germs

Suppose $p \in S(U)$ is C-definable, and f_a is an a-definable function defined on realisations of p.
Define the equivalence relation $\backsim$ as follows: $a \backsim a'$ iff for each $x \models p$, $f_a(x) = f_{a'}(x)$, i.e. $a \backsim a'$ iff $'f_a(x) = f_{a'}(x)' \in p$.
Then $\backsim$ is a C-definable equivalence relation. The $\backsim$-class of a is an imaginary, clled the <u>p-germ</u> of a.
Also, if f, g are defined in p, we that say f, g <u>have the same p-germ</u> if $'f(x) = g(x)' \in p$.

Definition. The p-germ e of f_a is strong over C, if there is a Ce-definable function g defined on p and with the same p-germ as f_a.

Fact. In a stable theory, any germ of a definable function on a definable global type is strong.

Theorem 16. *Let $C = acl(C)$ and $p \in S(U)$, with $p|C$ stably dominated. Let f_a be a definable function defined on realisations of p. Then*

(i) the p-germ of f_a is strong over C.

(ii) if $f(b) \in St_{Cb}$ for $b \models p$, then the p-germ of f is in St_C.

Remark. For (i), it suffices to show the following:
Let e be the p-germ of f. Then for any $a \equiv_{Ce} a'$,
$b \models p|Ca$ and $b \models p|Ca' \Rightarrow f_a(b) = f_{a'}(b)$. (Given this, $f_a(b) \in dcl(Ceb)$, and then we use compactness).
In the stable case, it is easy to show it: choose a'' with $a'' \downarrow_{Ce} aa'b$. Here it is complicated.

As a final point in this section, we just mention a generalisation of stable domination considered recently by Hrushovski and Pilllay (cf. [HP]).

Definition. A C-definable type $p \in S(U)$ is generically stable if it is finitely satisfiable in any small model $\mathcal{M} \supseteq C$.

Remark. If $p|C$ stably dominated (and p $Aut(\mathcal{U}/C)$-invariant), then p is generically stable.

6 Stable domination in ACVF, metastable theories

We will work in sorts $\mathcal{G}$ to ensure elimination of imaginaries.

Definition. Let $C = acl(C)$, $p \in S(C)$. Then $p \perp \Gamma$ if, for any $Aut(\mathcal{U}/C)$-invariant extension p' of p' over U, and any model $\mathcal{M}$, with $U \supseteq M \supseteq C$,
$a \models p'|M \Rightarrow dcl_\Gamma(Ma) = dcl_\Gamma(M)$ (here $dcl_\Gamma(X)$ means $dcl(X) \cap \Gamma$).

Remark.
- This definition is equivalent to the definition given in [HHM] (which needs $\downarrow^g$).
- $dcl_\Gamma(Ma) = dcl_\Gamma(M)$ means just that $tp(a/M)$ has a unique extension over Γ (by o-minimality of Γ).

Theorem 17. *(ACVF). Let $C = acl(C)$, $p \in S(C)$. The following are equivalent:*

(i) $p \perp \Gamma$.

(ii) p is stably dominated.

Remark. The generic type of a closed ball is stably dominated. The generic type of an open ball (or chain of balls with no least element) is not.

Example. If B is the open ball, and M is a model containing $\ulcorner B \urcorner$, then M contains a field element b of B. Now a field element a generic in B over M adds a new element $v(b-a) \in dcl_\Gamma(Ma) \setminus dcl_\Gamma(M)$ (for if $v(b-a) = \gamma \in \Gamma(M)$, than a is in the M-definable proper sub-ball $B_{\geq \gamma}$ of B, contradicting that a is generic in B).

Theorem 18 (Baur). *Let $(F, v) < (L, w)$ be an extension of valued fields, with F maximally complete. Let H be a finite dimensional F-subspace of L. Then H has a basis $h_1, \ldots, h_n$ such that for any $a_1, \ldots, a_n \in F$, $v(a_1u_1 + \ldots + a_nu_n) = Min\{v(a_iu_i) : 1 \leq i \leq n\}$.*

Theorem 19. *Let $\mathcal{M}$ be a maximally complete model of $ACVF$. Then for any a (in the sorts $\mathcal{G}$), $tp(a/M \cup dcl_\Gamma(Ma))$ is stably dominated.*

The key idea is to work over M. This gives a $*$-definable function $f = (f_i)$ to Γ with stably dominated fibres.
A special case of Theorem 19 is the following:

Proposition 4. *Let $\mathcal{M} \models ACVF$ be maximally complete and a be a sequence of field elements such that $dcl_\Gamma(M \cup a) = dcl_\Gamma(M)$. Then $tp(a/M)$ is stably dominated.*

Sketch of the proof. Let $A := \overline{M(a)}$, $B \models ACVF$ with $M \geq B$, and auppose $k(A)$ and $k(B)$ are linearly disjoint over $k(M)$ (i.e. $St_M(A) \downarrow_M St_M(B)$). We want to show that if $A' \equiv_B A$ and $k(A'), k(B)$ are linearly disjoint over M, then $A \equiv_B A'$.
Given $\bar{b}$ from B, we may write $\sum_{i=1}^n a_ib_i = \sum_{j=1}^l d_jb'_j$ such that

(i) $tp(d_1, \ldots, d_l/M)$ depends just on $tp(a_1, \ldots, a_n/M)$,

(ii) $tp(b'_1, \ldots, b'_l/M)$ depends just on $tp(b_1, \ldots, b_n/M)$,

(iii) $v(\sum_{i=1}^n a_ib_i) = \sum_{j=1}^l d_jb'_j = Min\{v(d_j + v(b'_j)) : 1 \leq j \leq l\}$.

Thus, if $(a_1, \ldots, a_n) \equiv_M (a'_1, \ldots, a'_n)$, then $v(\sum_{i=1}^n a_ib_i) = \sum_{i=1}^n a'_ib_i)$. By quantifier elimination, this suffices for $A \equiv_B A'$. □

6.1 Metastable theories

This part is based on recent work by Hrushovski (cf. [Hrus]).
We will say that a set is <u>$*$-definable</u>, if it is an ∞-definable set in (possibly) infinitely many variables. We can view it (e.g. interpreted in $\mathcal{U}$) as an inverse limit of definable sets.

Definition. Assume T is a complete multi-sorted theory, with a privileged sort Γ. Say T is metastable if:

(i) Γ is stably embedded,

(ii) no infinite definable subset of Γ^{eq} is stable,

(iii) any type over any $C = acl^{eq}(C)$ has a C-invariant extension,

(iv) for any partial type p over a base set C_0, there is $C_1 \supseteq C_0$ such that if $a \models p$, then $tp(a/C_1 \cup dcl_\Gamma(C_1 a))$ is stably dominated.

Remark.

1. Any o-minimal (or weakly o-minimal) theory is metastable (for stupid reasons).
2. Condition (iii) is there to ensure the 'descent' property of stable domination (i.e. decreasing the base preserves stable domination).
3. ACVF is metastable, with Γ as value group.
4. Also, $\mathbb{C}((t))$ and DCVF are metastable.
5. Sometimes, Hrushovski adds conditions (FD) or its strengthening $(FD)_\omega$.

   The condition (FD) means the following:

   - Γ is o-minimal,
   - for any definable set D, there is an upper bound on the Morley rank of definable images of D in St_C,
   - similar upper bound on o-minnimal dimension of definable images in Γ^{eq}.

We consider groups G which are definable (or ∞-definable, or $*$-definable) in a sufficiently saturated $\mathcal{U} \models T$, with elimination of imaginaries. We want to have a notion of 'generic type' in such groups. Once we have it, a generically metastable group will be one with a stably dominated generic type.
May be simplest to think of a generic metastable group, as one with a stably dominated translation-invariant global type.

Definition. Assume $p \in S(U)$ is a C-definable type of elements of G, where G is a definable (or ∞-definable, or $*$-definable) group.
If $a \in G$ then p has translates pa, pa, which are definable over $C \cup \{a\}$.
p is left generic if, for any $B = acl(B)$ over which p is defined, any right translate pa is B-definable. We define right generic similarly.

Fact. A stably dominated left generic type is right generic. So we just say "stably dominated generic type".

There is also a notion of symmetric type:
A type p is symmetric if, whenever q is a definable type, p, q both defined over C, then ($b \models q|C$ and $a \models p|acl(Cb)$) implies $b \models q|acl(Ca)$.
A symmetric left-generic type is right-generic. And a stably dominated lefgt generic type is symmetric.
As an example, the type at ∞ of $(\mathbb{R}, <, +)$ is left and right generic, but not symmetric.
With respect to left-generics the theory is nice:

- Any two symmetric left-generics differ by a left translation.
- G has an ∞-definable subgroup G^0 of bounded index with a unique symmetric left-generic.

Definition. G is generically metastable if it has a stably dominated (left) generic.

Example. In ACVF

- $(R, +)$ is generically metastable,
- $(K, +)$ is not.

Theorem 20 (Not under 'metastable theory' assumptions)**.** *Let G be a definable group with a translation-invariant stably dominated type p (stably dominated over C).*

(i) There is a $*$-definable (over C) stable group $\mathscr{G}$ and a $*$-definable (over C) group homomorphism $\phi : G \longrightarrow \mathcal{G}$ such that p is stably dominated over C by ϕ, i.e. if $a \models p$ and b is any tuple (with $\phi(a) \downarrow_C St_C(b)$), $tp(b/C \cup \phi(a)) \models tp(b/C \cup \{a\})$.

(ii) In ACVF, if $C = acl(C)$, then $\phi, \mathscr{G}$ can be chosen to be definable.

(ii) Given any other homomorphism $\phi' :\longrightarrow \mathcal{G}'$, with $\mathscr{G}'$ stable, there is $\psi : \mathscr{G} \longrightarrow \mathscr{G}'$ such that $\phi' = \psi \circ \phi$.

Sketch of the proof of (i). For any a realising p, let $\theta(a)$ enumerate $St_C(a)$. For each such a, define f_a on realisations of p, by $f_a = \theta(ab)$. As $f_a(b) \in St_C$, the p-germ e of f_a is in St_C. As e is Ca-definable, $e \in \theta(a)$. As e is a strng germ, there is $f'_{\theta(a)}$, defined over $\theta(a)$, with the same p-germ as f_a. Let $b \models p|Ca$, $d = f_a(b)$. As St_C is stably embedded, and $d \in St_C$, $tp(\theta(a), d/C\theta(b)) \vdash tp(\theta(a), d/Cb)$. (NB: $\theta(a), d$ are in St_C).
So as $d \in dcl(C, \theta(a), b)$, $d \in dcl(C, \theta(a), \theta(b))$. Then we have a C-definable function $*$-definable function F, with $d = F(\theta(a), \theta(b))$, i.e. $\theta(ab) = F(\theta(a), \theta(b))$. F is generically associative and has the other required properties. Then, by a group chunk theorem, it is the restriction of a group operation ($*$-definable). And θ is generically a homomorphism, so it is the restriction of a group homomorphism. □

We can give as examples:

- $G = (R, +)$ and $\mathscr{G} = (k, +)$
- $G = SL_n(R)$ and $\mathscr{G} = SL_n(k)$

6.2 Other results on metastable groups

Theorem 21. *Let G be a generically metastable group definable in $K \models ACVF$. Then there is an algebraic group H over K and a definable homomorphism $f : G \longrightarrow H(K)$ such that $Ker(f)$ is 'boundedly imaginary', i.e. there is no definable map from $Ker(f)$ to an unbounded subset of Γ.*

Theorem 22 (ACVF)**.** *Let H be an affine algebraic group, G a generically metastable definable subgroup . Then for some algebraic group scheme H_1 over R, G is isomorphic to $H_1(R)$.*

Theorem 23. *Let T be a metastable theory with satisfiying $(FD)_\omega$, and let A be a definable abelian group. Then there is a definable group Λ in Γ^{eq}, and a definable homomorphism $\lambda : A \longrightarrow \Lambda$ such that $Ker(\lambda)$ is 'limit metastable', i.e. is a direct limit (with respect to a $*$-definable directed system) of metastable groups.*

NB: $(K, +)$ is limit metastable.

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