NOTES ON "GEOMETRIC MOTIVIC INTEGRATION"

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This note is a supplement to the lecture "Geometric motivic integration", given by Johannes Nicaise [Nic08]. As the lecture (and the slides) are mostly self-contained, we shall address a point that did not appear directly as part of the lecture, namely, we will sketch a more model theoretic description of the arc spaces and some other concepts that appear in the lecture. We will also make some explicit calculations with the change of variable formula.

1. Arc spaces as definable sets

Motivic integration is presented in the lecture from a purely algebraic point of view: the jet spaces $\mathcal{L}_n(X)$ and the arc space $\mathcal{L}(X)$ are schemes, The measurable sets are defined in terms of certain constructible subsets (in the algebraic sense) of these schemes (and more generally, subsets approximated by such sets.) There is an alternative description in terms of definable sets in theories of valued fields. This is the way motivic integration is described in the theories of Loeser–Cluckers [CL08] and of Hrushovski–Kazhdan [HK06], but this approach appears already in the theory of Denef–Loeser (c.f. [Loe03], for example, for an overview.)

In this approach, one works in a theory T of valued fields of residue characteristic 0. There are various possible choices for T, for example:

- The theory DP of $k[[t]]$ in the Denef–Pas language. This language includes sorts for the valued field $\mathbb{K}$, the residue field $\Bbbk$ and the value group Γ (as an ordered abelian group), with symbols for the valuation function, and an *angular component* function: this is a group homomorphism from $\mathbb{K}^*$ to $\Bbbk^*$ whose restriction to $\mathcal{O}^*$ is the residue map.
- The theory $ACVF_0$ of algebraically closed valued fields (of characteristic 0.) The language here includes, in addition to the valued field $\mathbb{K}$ and the value group Γ, a sort RV for $\mathbb{K}^*/(1+\mathfrak{m})$, ($\mathfrak{m}$ is the maximal ideal) with all the induced structure, and the quotient map to Γ. This is the language used in [HK06]. Note that $\Bbbk^*$, the multiplicative group of the residue field of $\mathbb{K}$ is included as a definable subset of RV (As pointed out in [HK06, section 12], this example includes the previous one, but for the sake of presentation it is convenient to think of them as distinct examples.)

Whatever theory T is used, the principles outlined below remain the same. First, T includes at least two sorts, $\mathbb{K}$ for the valued field, and $\Bbbk$ for the residue field (with at least constants in $\Bbbk$ for the base field k), and for any field extension A of k, $A((x))$ with the usual valuation generates a structure for T.

Varieties over k are identified (after choosing a local affine presentation) with the corresponding definable sets in $\Bbbk$. If X is such a variety, then (at least if X is smooth) $\mathcal{L}(X)$ is identified with the definable subset of $\mathbb{K}$ defined by the same equations, intersected with the closed ball of radius 1 around 0 (the valuation ring.) Indeed, any quantifier free definable set Y in $\mathbb{K}^n$ determines a functor $A \mapsto$

$A[[t]] \mapsto Y(A[[t]])$, and if Y is a variety, this is precisely the definition of $\mathcal{L}(Y)$ (the smoothness assumption is required since the above functor is defined only on fields A, hence only the reduced structure of $\mathcal{L}(Y)$ is determined.) If $Z \subseteq Y$ is a subvariety, the definable subset $res(\bar{y}) \in Z$ of $\mathcal{L}(Y)$ is, by definition, a cylinder.

The point now is that any definable subset of $\mathbb{K}^n$ is a boolean combination of cylinders and sub-varieties (which have measure 0.) This follows from quantifier elimination for T in each case (at least in the valued field sort), since this can be checked directly for the quantifier free sets. Hence any such definable subset is measurable. We note that not every cylinder is definable: for example, the subset of points $a_0 + a_1 t + \cdots \in \mathcal{L}(\mathbb{A}^1)(k)$ consisting of points where $a_1 = 0$ is a cylinder (the inverse image of $\mathbb{A}^1 \times \{0\} \subseteq \mathbb{A}^2 = \mathcal{L}_1(\mathbb{A}^1)$), but not a definable subset. However, most measurable sets that occur in applications are, in fact, definable. In particular, if α is some definable function from $\mathbb{K}$ to Γ, then its fibres are also definable, and so α is integrable (at least if its image is bounded.)

Furthermore, it turns out that when the measure function is restricted to definable sets, it can be described axiomatically: it is the unique function μ from definable subsets of $\mathbb{K}$ to the (localised) Grothendieck group of the residue field (or, more generally, of some of the other sorts), that satisfies some natural properties expected of measures. The precise axiomatisation of course depends on the particular theory, but the properties may include:

(1) If Y has the form $res(\bar{x}) \in Z$, where Z is a definable subset of $\Bbbk^d$, then $\mu(Y) = [Z]/\mathbb{L}^d$. Here $\mathbb{L} = [\mathbb{A}^1]$ is the class of the affine line.
(2) $\mu(Y_1 \times Y_2) = \mu(Y_1)\mu(Y_2)$
(3) μ is invariant under translations, and commutes with disjoint unions.
(4) $\mu(val(x) \geq i) = \mathbb{L}^{-i}$.

Thus, from this point of view, motivic integration is simply a detailed study of definable sets in theories of valued fields, and especially the relation between classes of definable sets in the valued field sort with classes of definable sets in the other sorts (residue field, value group, etc.)

2. Some calculations

In this section, we illustrate the change of variables formula, by computing two examples. Since the main point is the contribution of the Jacobian, and since the integrals of functions is determined by the measure of subsets, we shall restrict attention to computing the measure of a space, rather than the integral of a general function. In other words, we apply the formula when $\alpha = 0$. On the other hand, we shall ignore the assumption that $val(Jac_h)$ takes only finitely many values, since it is rather unrealistic, and is only needed to avoid introducing convergent sums in $\mathcal{M}_k$.

We remark that geometrically, if I is the ideal (sheaf) that corresponds to a subvariety V of X, then the set $(ord_t I)(x) \geq i$ is the set of points that "approximately lie in V" (up to approximation of order i.) This should not be confused, as will be shown in the second example, with the set of points of distance i from V. In any case, the set is definable, and thus measurable.

2.1. **Blow up of affine space at 0.** We apply the theorem when $X = \mathbb{A}^2$ and $h : Y \to X$ is its blow-up at 0. We should get that $1 = \mu(\mathbb{A}^2)$ is equal to

$$\int_{\mathcal{L}(Y)} \mathbb{L}^{-val(Jac_h)} = \sum_{i \geq 0} \mu((ord_t Jac_h)(x) = i)\mathbb{L}^{-i} \tag{1}$$

To compute the sum, we first note that Jac_h is simply the ideal corresponding to the exceptional divisor (cf. [Har77, ex. II.8.5], or compute directly) which in this case is $\mathbb{P}^1$. If we set $B_i = \mu(ord_t Jac_h(x) \geq i)$, we need to compute $\sum_{i \geq 0}(\mu(B_i) - \mu(B_{i+1}))\mathbb{L}^{-i}$. However, B_{i+1} itself is already explicitly a (stable) cylinder: it is simply the inverse image of $\pi_i(\mathcal{L}(\mathbb{P}^1)) = \mathcal{L}_i(\mathbb{P}^1)$, and the last is equal in the Grothendieck ring to $[\mathbb{P}^1]\mathbb{L}^i$, since it is smooth and of dimension 1. Gathering everything, we get:

$$\sum_{i \geq 0}(\mu(B_i) - \mu(B_{i+1}))\mathbb{L}^{-i} = \mu(\mathcal{L}(Y)) - \mu(\pi^{-1}(\mathbb{P}^1)) + \sum_{i \geq 1}\left(\frac{[\mathbb{P}^1]\mathbb{L}^{i-1}}{\mathbb{L}^{2i}} - \frac{[\mathbb{P}^1]\mathbb{L}^i}{\mathbb{L}^{2i+2}}\right)\mathbb{L}^{-i} =$$

$$= \frac{\mathbb{L}^2 - 1}{\mathbb{L}^2} + \frac{[\mathbb{P}^1](\mathbb{L} - 1)}{\mathbb{L}^2} \sum_{i \geq 1} \frac{1}{\mathbb{L}^{2i}} = \frac{\mathbb{L}^2 - 1}{\mathbb{L}^2} + \frac{(\mathbb{L} + 1)(\mathbb{L} - 1)}{\mathbb{L}^2} \frac{1}{\mathbb{L}^2 - 1} = 1$$

The last few steps included the following observations: First, $\mathcal{L}(Y) - \pi^{-1}(\mathbb{P}^1)$ is the same as $\pi^{-1}(Y - \mathbb{P}^1)$ and $[Y] - [\mathbb{P}^1] = [X] - 1 = \mathbb{L}^2 - 1$. Second, the class of $\mathbb{P}^1$ itself is $\mathbb{L} + 1$. Finally, we can compute the infinite sum above using the usual formula if we agree that "$\mathbb{L} > 1$" (it turns out that if one only computes the measures of definable sets, this is one of very few infinite summations that one needs to perform.) Thus equation (1) is verified. We remark that essentially the same calculation applies to an arbitrary blow-up of a non-singular variety along a non-singular subvariety (of codimension 2.)

We now try to give some intuition why the formula should hold. Identify $\mathcal{L}(X)$ with the definable set $\mathbb{K}^2$ in the valued field, and let B_X be the open ball of (valuative) radius 1 around the point (t, t). Thus, B_X is the definable set given by $val(x - t) > 1$ and $val(y - t) > 1$. Over B_X, we may view Y as given by the equation $zx = y$ in 3-space. The set B_X is thus approximated in Y by the formulae $val(x - t) > 1$, $val(y - t) > 1$ and $val(y - zx) > 1$. We now compute the measures of these sets using the axioms from the previous section. Hence B_X has measure $\mu(val(x-t) > 1)\mu(val(y-t) > 1) = \mu(val(x) > 1)\mu(val(y) > 1) = \frac{1}{\mathbb{L}^4}$. On the other hand, the set B_Y can also be defined by replacing the formula $val(y - zx) > 1$ with $val(1 - z) > 0$. Thus we may invoke the same rules again to compute the measure of B_Y, and we get another factor of $\frac{1}{\mathbb{L}}$. In general, if we vary the radius of the various balls, we get from the formula $val(z) + val(x) = val(y)$ a factor of $\frac{1}{\mathbb{L}^{val(r_x)}}$, where r_x is the radius of the ball around x. This corresponds to the fact that the Jacobian, in this case, is generated by x. Thus the balls are "squeezed" by the value of the Jacobian.

2.2. **Blow-up of a singular curve.** Let X be the affine curve given by the equation $x^2 = y^3$ in $\mathbb{A}^2$ (this is the example that appeared in the lecture.) Let $Y = \mathbb{A}^1 = spec(k[z])$ and let $h : Y \to X$ be the map $z \mapsto (z^3, z^2)$ (this is again a blow up: z is the slope at its image.) This time, we don't know what to expect, since X is singular. Using the change of variables formula we get:

$$\mu(X) \quad = \quad \sum_{i\geq 0}(\mu(B_i) \; - \; \mu(B_{i+1}))\mathbb{L}^{-i} \quad = \quad (\frac{1}{\mathbb{L}^i} \; - \; \frac{1}{\mathbb{L}^{i+1}})\mathbb{L}^{-i} \quad = \quad \frac{\mathbb{L}}{\mathbb{L}+1}$$

Here, the exceptional divisor is just one point, and B_i is the closed ball of radius i around it.

We could also try to compute the measure in another way: If we remove the origin, we are left with a smooth curve. Thus, a "first approximation" to the measure of X is $\frac{[X]-1}{\mathbb{L}}$. To reduce the error, we may remove the origin in the approximation of $[X]$ in a higher $\mathcal{L}_n$. Thus the second approximation is obtained by removing the variety given by $x_0 = y_0 = 0$ in the equations for $\mathcal{L}_1(X)$. The equations for $\mathcal{L}_1(X)$ are ${x_0}^2 - {y_0}^3 = 0$ and $2x_0x_1 - 3{y_0}^2y_1 = 0$. Thus the next approximation appears to be

$$\frac{([X]-1)\mathbb{L}+\mathbb{L}^2-1}{\mathbb{L}^2} = \frac{[X]-1}{\mathbb{L}} + 1 - \frac{1}{\mathbb{L}^2}$$

In the third approximation we add the equation ${x_1}^2 + 2x_0x_2 - 3y_0{y_1}^2 - 3{y_0}^2y_2 = 0$ (these equations were presented in the lecture), and get

$$\frac{([X]-1)\mathbb{L}^2+\mathbb{L}^3-1}{\mathbb{L}^3} = \frac{[X]-1}{\mathbb{L}} + 1 - \frac{1}{\mathbb{L}^3}$$

Continuing this way, we get a convergent sequence, that converges to $\frac{[X]-1}{\mathbb{L}} + 1$. However, the map h above is a bijection between X and $\mathbb{A}^1$, so $[X] = \mathbb{L}$ and we see that the answer is different! Where is the mistake?

The process just described is in fact similar to the definition of the measure in the singular case. However, the approximations should be replaced by taking neighbourhoods of actual solutions, rather than just approximate solutions. The problem is that when X is singular, the projections between the $\mathcal{L}_i$ need not be *surjective*. Indeed, the image of the projection from $\mathcal{L}_2$ to $\mathcal{L}_1$ in the above example does not include any point in the tangent space at 0 with $x_1 \neq 0$. Furthermore, the equation $x^2 = y^3$ itself forces that $2val(x) = 3val(y)$, so we immediately see that if $x_0 = y_0 = 0$, then so are x_1, x_2 and y_1. Thus in this case the change of variables formula really helps!

References

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