

Dependent Fields have no Artin-Schreier Extensions

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Introduction

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In [5], Macintyre showed that an infinite ω -stable field is algebraically closed; this was subsequently generalized by Cherlin and Shelah to the superstable case, and commutativity need not be assumed but follows [3]. It is known [6] that separably closed fields are stable; the converse has been conjectured early on, but little progress has been made. In 1992, Thomas Scanlon noted that an infinite stable field of characteristic p at least has no Artin-Schreier extensions, and hence has no finite Galois extension of degree divisible by p . Here we show the same for dependent (NIP) infinite fields.

Notation

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Notation is standard.

Dependent theories

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Definition

A theory T has the **independence property** if there is a formula $\phi(\bar{x}, \bar{y})$ and tuples $(\bar{a}_i : i \in \omega)$ and $(\bar{b}_A : A \subset \omega)$ such that $\models \phi(\bar{a}_i, \bar{b}_A)$ if and only if $i \in A$.

Definition

A theory T is **dependent** if it does not have the independence property (also known as **NIP**).

Example

Every stable theory is NIP as well as any o-minimal (and many others).

Dependent groups

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Definition

A group G is dependent if $Th(G)$ is dependent.

We use only the following theorem about dependent groups:

Theorem (Baldwin-Saxl)

[1] *Let G be a dependent group. Given a family of uniformly defined subgroups, there is a number $n < \omega$ such that any finite intersection of groups from this family is already an intersection of n of them.*

Artin-Schreier theory

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Let k be a field of characteristic p . Let ρ be the polynomial $X^p - X$.

Theorem (Artin-Schreier)

- 1** *Given $a \in k$, either the polynomial $\rho - a$ has a root in k , in which case all its roots are in k , or it is irreducible. In the latter case, if α is a root then $k(\alpha)$ is cyclic of degree p over k .*
- 2** *Conversely, let K be a cyclic extension of k of degree p . Then there exists $\alpha \in K$ such that $K = k(\alpha)$ and for some $a \in k$, $\rho(\alpha) = a$.*

Such extensions are called Artin-Schreier extensions.

Main Theorem

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Theorem

*Let K be an infinite dependent field of characteristic $p > 0$.
Then K is Artin-Schreier closed - i.e. ρ is onto.*

Corollary

*If L/K is a finite separable extension, then p does not divide
 $[L : K]$.*

Corollary

K contains \mathbb{F}_p^{alg} .

Beginning of proof

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- First we work in a more general setting. Let F be an algebraically closed field
- For a number $n < \omega$ and $\bar{b} \in F^{n+1}$, we define the algebraic group

$$G_{\bar{b}} = \{(t, x_1, \dots, x_n) \mid t = b_i(x_i^p - x_i) \text{ for } 1 \leq i \leq n\}.$$

- $G_{\bar{b}}$ is an algebraic subgroup of $(K, +)^{n+1}$.
- If $\bar{b} \in K \subseteq F$, then by Baldwin-Saxl, for some number n_0 , for every $n_0 + 1$ tuple \bar{b} , there is a sub n_0 tuple \bar{b}' such that the projection $\pi : G_{\bar{b}}(K) \rightarrow G_{\bar{b}'}(K)$ is onto. (Consider the family of subgroups of the form $\{t \mid \exists x [t = a(x^p - x)]\}$).

First claim

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Claim

Let F be an algebraically closed field. Suppose $\bar{b} \in F^\times$ is algebraically independent, then $G_{\bar{b}}$ is a connected group.

Proof.

(sketch) By induction on $n := \text{length}(\bar{a})$. For $n = 1$ it's easy. Assume the claim for n , and let $\text{length}(\bar{a}) = n + 1$, $\bar{a}' = \bar{a} \upharpoonright n$. $\pi : G_{\bar{b}} \rightarrow G_{\bar{b}'}$ is onto. Let $H = G_{\bar{b}}^0$ (the connected component of 0). $[G_{\bar{b}'} : \pi(H)] \leq [G_{\bar{b}} : H] < \infty$, so $\pi(H) = G_{\bar{b}'}$ by induction. Assume that $H \neq G_{\bar{b}}$. Then for every $(t, \bar{x}) \in G_{\bar{b}'}$, there is exactly one x_{n+1} such that $(t, \bar{x}, x_{n+1}) \in H$.

First claim (cont)

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Proof.

So H is a graph of a definable function over \bar{b} . Now choose x_i such that $b_i \cdot (x_i^p - x_i) = 1$ for $i = 1, \dots, n$. Then we can find a solution to $b_{n+1} \cdot (x^p - x) = 1$ in

$dcl(b_{n+1}, x_1, \dots, x_n) = \mathbb{F}_p(b_{n+1}, x_1, \dots, x_n)_{ins}$. (for a field L , $L_{ins} = \bigcup_{n < \omega} L^{p^{-n}}$). Denote $L = \mathbb{F}_p(b_{n+1}, x_1, \dots, x_n)$. As x is separable over L , it follows that $x \in L$. By our assumption, b_{n+1} is transcendental over x_1, \dots, x_n . From this one can easily derive a contradiction. \square

Second claim

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Claim

Let k be a field of characteristic p , and let $f : k \rightarrow k$ be an additive polynomial (i.e. $f(x + y) = f(x) + f(y)$). Then f is of the form $\sum a_i x^{p^i}$.

Moreover, if k is algebraically closed and $\ker(f) = \mathbb{F}_p$ then $f = (a \cdot (x^p - x))^{p^n}$ for some $n < \omega$, $a \in k^\times$.

Proof.

(sketch) By induction on $\deg(f)$. First show that $f'(x)$ is constant (a_0), and prove the first part. The induction base for the moreover: if $a_0 \neq 0$, then $(f, f') = 1$, hence f has no multiple factors, hence $\deg(f) = p$, hence $f = a_0 x + a_1 x^p$, but letting $x = 1$ we get $a_0 = -a_1$. \square

Important fact

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We use the following:

Fact

Let k be a perfect field, and G a closed 1-dimensional connected algebraic subgroup of $(k^{\text{alg}}, +)^n$ defined over k , for some $n < \omega$. Then G is isomorphic over k to $(k^{\text{alg}}, +)$.

The proof of this fact uses Hilbert 90, and the following, from [4]

Fact

A closed connected subgroup of a $(k^{\text{alg}}, +)^n$ group is isomorphic to $(k^{\text{alg}}, +)^{\dim(G)}$.

Proof continues

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Let us use the fact and the 2 claims to finish the proof.

- We may assume that K is \aleph_0 saturated.
- Let $k = \bigcap_n K^{p^n}$. k is an infinite perfect field.
- choose $\bar{b} \in k^{n_0+1}$ that are algebraically independent.
- By Baldwin-Saxl, there is some sub sub n_0 tuple \bar{b}' such that the projection $\pi : G_{\bar{b}}(K) \rightarrow G_{\bar{b}'}(K)$ is onto.
- By the first claim, both $G_{\bar{b}}$ and $G_{\bar{b}'}$ are connected. Of course, their dimension is 1.

Proof continues

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- By the fact, we know that both these groups are isomorphic over k to $(K^{alg}, +)$.
- So we have some $\nu \in k[X]$ such that

$$\begin{array}{ccc} G_{\bar{b}}(K^{alg}) & \xrightarrow{\pi} & G_{\bar{b}'}(K^{alg}) \\ \downarrow & & \downarrow \\ (K^{alg}, +) & \xrightarrow{\nu} & (K^{alg}, +) \end{array}$$

commutes.

- As the sides are isomorphisms, defined over $k \subseteq K$, we can restrict them to K . As $\pi \upharpoonright G_{\bar{b}}(K)$ is onto $G_{\bar{b}'}(K)$, then so is $\nu \upharpoonright K$.
- $|\ker(\nu)| = p = |\ker(\pi)|$ (even when restricted to K).

Proof ends

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- Suppose that $0 \neq c \in \ker(\nu) \subseteq k$. let $\nu' = \nu \circ m_c$
($m_c(x) = c \cdot x$).
- ν' is an additive polynomial over K whose kernel is \mathbb{F}_p . So WLOG $\ker(\nu) = \mathbb{F}_p$.
- By the second claim, ν is of the form $a \cdot (x^p - x)^{p^n}$ for $a \in K^\times$.
- But ν is onto, hence so is ρ and we are done (given $y \in K$, there is some $x \in K$ such that $a \cdot (x^p - x)^{p^n} = a \cdot y^{p^n}$)

On Simple fields

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It is known that bounded (i.e. with only finitely many extensions of each degree) PAC (pseudo-algebraically closed: every absolutely irreducible variety has a rational point) fields are simple [2] and again the converse is conjectured. In 2006, Frank Wagner adapted Scanlon's argument to the simple case and showed that simple fields have only finitely many Artin-Schreier extensions. He even showed this is true when the field is type definable in a simple theory.

type definable fields

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As of yet, I do not know the situation with type definable fields in a dependent theory. What is known is that

- For a field type definable in a dependent theory there are either infinitely many Artin-Schreier extensions, or none.

References

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