Modnet tutorial at La Roche. I. Introduction to functional analogues of the Lindemann-Weierstrass theorem.

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- The aim of the subseries by Bertrand and Pillay is to both state as well as give some ingredients of the proof of an analogue of what is called Lindemann's theorem (or Lindemann-Weierstrass) on transcendence properties of exponentials of algebraic numbers, for certain commutative algebraic groups over $\mathbb{C}(t)$ (or the algebraic closure of $\mathbb{C}(t)$).
- Model theory, or at least some model theory of differentially closed fields, plays a role in the proof.
- There is some fascinating work of Boris Zilber around Schanuel's conjecture (a strengthening of Lindemann's theorem) and the model theory of complex exponentiation. But this is not the topic of this subseries, although it may touched on by other speakers.
- To state our result in full generality requires introducing a number of objects and concepts, such as semiabelian varieties, algebraic D-groups, generalized logarithmic derivatives,....
- Daniel's talks will include a more algebraic-geometric account of some of these objects.

Lindemann-Weierstrass I

- This classical result says that if $x_1, .., x_n \in \mathbb{Q}^{alg}$ (the algebraic closure of $\mathbb{Q}$) are $\mathbb{Q}$-linearly independent, then the complex numbers $e^{x_1}, ..., e^{x_n}$ are algebraically independent, that is $tr.deg(\mathbb{Q}(e^{x_1}, .., e^{x_n})/\mathbb{Q}) = n$.
- Note that the map taking $(x_1, .., x_n) \in \mathbb{C}^n$ to $(e^{x_1}, .., e^{x_n}) \in \mathbb{C}^n$, is a surjective complex-analytic homomorphism exp from the complex algebraic group $(\mathbb{C}^n, +)$ to the complex algebraic group $((\mathbb{C}^{\times})^n, \times)$. The former is the Lie algebra of the latter, and this exp makes sense for any commutative algebraic group and its Lie algebra.
- Note that the algebraic subgroups of $(\mathbb{C}^{\times})^n$ are defined by systems of equations: $y_1^{k_1} \cdot .... \cdot y_n^{k_n} = 1$.
- Moreover if H is defined by such an equation then LH is defined by $k_1 x_1 + ... + k_n x_n = 0$.

Lindemann-Weierstrass II

- Hence the $\mathbb{Q}$-linearity hypothesis in Lindemann's theorem can be restated as: $x = (x_1, .., x_n) \notin L(H)$ for every proper algebraic subgroup H of $(\mathbb{C}^\times)^n$.

Functions in place of numbers I

- We want to consider a simultaneous analogue of L-W in two directions (a) with "functions" $x_1(t), .., x_n(t)$ in place of algebraic numbers, and (b) with other commutative algebraic groups, even defined over $\mathbb{C}(t)^{alg}$, in place of $((\mathbb{C}^\times)^n, \times)$.
- In standard notation, which should make a lot of sense to model-theorists, $(\mathbb{C}^n, +) = \mathbb{G}_a^n(\mathbb{C})$, and $((\mathbb{C}^\times)^n, \times) = \mathbb{G}_m^n(\mathbb{C})$.
- We first consider (a) for these groups, i.e. for algebraic tori and their Lie algebras.
- If $K = \mathbb{C}(t)$, then a point $x = (x_1, .., x_n) \in \mathbb{G}_a^n(K)$ is simply an n-tuple of rational functions $(x_1(t), .., x_n(t))$ and then $exp(x)$ is an n-tuple of meromorphic functions (or meromorphic functions on some open set) e^{x_i} where $e^{x_i}(t) = e^{x_i(t)}$.
- We are interested in the transcendence degree of $K(exp(x)/K)$.

Functions in place of numbers II

- The natural hypothesis is that $x_1, .., x_n \in K$ are $\mathbb{Q}$-linearly independent modulo $\mathbb{C}$, which is equivalent to:
- There is no proper algebraic subgroup H of $\mathbb{G}_m^n$ such that $x \in LH(K) + \mathbb{G}_a^n(\mathbb{C})$.
- The conclusion should be that $tr.deg(K(exp(x))/K) = n$, namely that e^{x_1},..,e^{x_n} are algebraically independent over K.
- This functional L-W statement is TRUE, i.e. the conclusion does follow from the hypothesis.
- Moreover the result follows from (or is equivalent to) a differential algebraic statement which we briefly explain.
- Let $K = \mathbb{C}(t)$ and $L > K$ a field of meromorphic functions containing $exp(x)$, and make it a differential field by putting $\partial = d/dt$.
- Note that $\partial(e^{x_i}) = e^{x_i}\partial(x_i)$.

Functions in place of numbers III

- Hence e^{x_i} is a solution of $\partial(y_i)/y_i = \partial(x_i)$.
- The map taking $(y_1, .., y_n) \in \mathbb{G}_m^n(L)$ to $(\partial(y_1)/y_1, ..., \partial(y_n)/y_n) \in \mathbb{G}_a^n(L)$ is the standard "logarithmic derivative" on $\mathbb{G}_m^n(L)$, written as $\partial ln_{\mathbb{G}_m^n}$.
- So the functional L-W theorem follows from the statement (special case of Ax's theorem, but also proved by Kolchin):
- Suppose (L, ∂) is a differential field extending $(K, d/dt)$, $x \in \mathbb{G}_a^n(K)$ is such that $x \notin LH + \mathbb{G}_a(\mathbb{C})$ for any proper algebraic subgroup H of $\mathbb{G}_m^n$. Suppose that $y \in \mathbb{G}_m^n(L)$ satisfies $\partial ln_{\mathbb{G}_m^n}(y) = \partial(x)$. THEN $tr.deg(K(y)/K) = n$.

Constant and nonconstant groups I

- We now bring in the other aspect (b) of our desired generalization.
- The first level of generality is to consider arbitrary complex commutative algebraic groups G together with their Lie algebra (tangent space at identity) LG, the exponential map $exp : LG(\mathbb{C}) \to G(\mathbb{C})$, as well as the appropriate logarithmic derivative ∂ln_G from $LG(F) \to G(F)$ for any differential field containing $\mathbb{C}$.
- Here exp_G is the unique analytic homomorphism from $LG(\mathbb{C}) \to G(\mathbb{C})$ whose differential at 0 is the identity $id_{LG(\mathbb{C})}$. The appropriate logarithmic derivative, defined by Kolchin, again gives the differential equations satisfied by exp.

Constant and nonconstant groups II

- Among the new groups entering the picture are *abelian varieties*: commutative complex algebraic groups whose underlying variety is projective. As complex Lie groups they are *compact*.
- A general commutative algebraic group G fits into a short exact sequence:
- $0 \to L \to G \to A$ where A is an abelian variety, and L is a the direct product of a "vector group" $\mathbb{G}_a^{n_1}$ and an "algebraic torus" $\mathbb{G}_m^{n_2}$.
- The straight analogue of L-W for powers of an elliptic curve defined over $\mathbb{Q}$ with complex multiplication is known but not much more.
- The functional analogue of L-W (in the differential version) for commutative algebraic groups defined over $\mathbb{C}$ and with no vectorial quotients, is again true. (Ax, Bertrand,..)

Constant and nonconstant groups III

- There are infinite "moduli spaces" for abelian varieties, that is there are really moving families of abelian varieties, in contrast to linear commutative algebraic groups. For example, an elliptic curve is determined up to isomorphism, by its j-invariant which can be any element of the underlying field.
- So this immediately suggests trying to formulate and prove functional analogues of L-W where the relevant commutative algebraic group is defined over say $\mathbb{C}(t)$ or its algebraic closure, and is not necessarily isomorphic to a group defined over $\mathbb{C}$.
- An (commutative, connected) algebraic group G say, defined over $K = \mathbb{C}(t)$ can be viewed as the "generic fibre" of an algebraic family of complex algebraic groups G_t parametrized by the affine line over $\mathbb{C}$.

Constant and nonconstant groups IV

- Likewise, if it is defined over $K = \mathbb{C}(S)$ for S a curve then we are talking about a family parametrized by S or at least an open subset.

Constant and nonconstant groups V

- In any case we have an algebraic family $\mathbf{G}(\mathbb{C}) \to S(\mathbb{C})$ of commutative complex algebraic groups G_s say, and the "generic fibre" is G. Can we again make sense of $exp_G(x)$ for $x \in G(K)$?
- For $s \in S$ we have $exp_{G_s} : LG_s(\mathbb{C}) \to G_s(\mathbb{C})$.
- $x \in G(K)$ "is" or gives rise to a rational section $x : S(\mathbb{C}) \to L(\mathbf{G})(\mathbb{C})$.
- For a suitable small open $U \subseteq S$ (in Euclidean topology), x gives rise to an analytic section $x : U \to L(\mathbf{G})_U$, and by $exp_G(x)$ we mean the analytic section $U \to \mathbf{G}_U$ which takes s to $exp_{G_s}(x(s))$.
- Note that $exp_G(x)$ lies in the field L of meromorphic functions on U (which naturally contains K)
- Before stating our main theorem, we will give a corollary of it in the language of this slide:

Constant and nonconstant groups VI

Theorem 1.1

Suppose that A is an abelian variety of dimension n over $K = \mathbb{C}(t)^{alg}$. Let A_0 be the $\mathbb{C}$-trace of A (explain). Let $x \in LA(K)$ and suppose
$\mathbf{HX}_K$: $x \notin L(B)(K) + L(A_0)(\mathbb{C})$ for any proper abelian subvariety B of A.
Let $y = exp_A(x)$. Then $tr.deg(K(y)/K) = n$.

Note that when the $\mathbb{C}$-trace of A is 0 then the hypothesis $\mathbf{HX}_K$ is simply that $x \notin L(B)$ for any proper algebraic subgroup B of A, and we obtain a statement very similar to the arithmetic L-W theorem.
The question of what differential equations are satisfied by these "relative" exponential maps is a delicate one, related to Picard-Fuchs equations, Gauss-Manin connection,..., some of which will be discussed in Daniel's talks.

Algebraic D-groups and generalized logarithmic derivatives I

- In this section we discuss the objects necessary for even stating our main theorem
- We let (K, ∂) be a differential field (say algebraically closed and of characteristic 0), which you can consider as embedded in a big saturated differentially closed field $(\mathcal{U}, \partial)$. We may sometimes take K to be $\mathbb{C}(t)^{alg}$.
- Let G be a connected commutative algebraic group over K. A D-group structure on G (defined over K) can be defined in a couple of ways:
- The first is as an extension of the derivation ∂ to a derivation ∂' of the structure sheaf $\mathcal{O}_{\mathcal{G}}$ (or coordinate ring for affine groups) of G over K which respects the group operation. Explain.

Algebraic D-groups and generalized logarithmic derivatives II

- The second is as a rational homomorphic section $s : G \to T_\partial(G)$, where $T_\partial(G)$ is the "twisted tangent bundle" of G, itself a commutative algebraic group.
- $T_\partial(G)$ is defined (locally) by equations $P(x_1, .., x_n)$ together with $\sum_i \partial P/\partial x_i(x_1, , .x_n)u_i + P^\partial(x_1, ., x_n) = 0$ for P polynomials over K defining G.
- If f is a system of polynomials defining (locally) the group operation on G then the group operation on $T_\partial(G)$ is given by $(g, u) \cdot (h, v) = (g \cdot h, df_{(g,h)}(u, v) + f^\partial(g, h))$.
- In the case where say G is defined over $\mathbb{C}(t)^{alg}$, and so is the generic fibre of a fibration $\mathbf{G}(\mathbb{C}) \to S(\mathbb{C})$, then we obtain $T(\mathbf{G}) \to T(S)$, and for s a generic point of S over $\mathbb{C}$, $T_\partial(G)$ corresponds to the fibre of $T(\mathbf{G})$ over $(s, \partial(s))$, and so can also be described as the image of the vector field ∂ on S under the Kodaira-Spencer map, at the generic point of S.

Algebraic D-groups and generalized logarithmic derivatives III

- If G has no nontrivial homomorphisms to $\mathbb{G}_a$ (so if G is semiabelian, or the universal vector extension of a semiabelian variety) then G has at most one structure of an algebraic D-group.
- Buium proves that a semiabelian variety has a structure of a D-group if and only if G descends to the constants of K, in which case $s = 0$ is the unique D-structure.
- On the other hand the "universal extension by a vector group" $\tilde{G}$ of any semiabelian variety G has a (unique) structure of D-group.
- A D-group structure s on a commutative algebraic G yields three additional objects, (i) $(G, s)^{\partial}$, the subgroup of "horizontal" elements, (ii) a generalized logarithmic derivative from G to LG, and (iii) a "connection" on LG.

Algebraic D-groups and generalized logarithmic derivatives IV

- We work with $G(\mathcal{U})$. Given $g \in G(\mathcal{U})$, we can apply the derivation ∂ to the coordinates of g to obtain $\partial(g) \in T_{\partial}(G)_g$. Then $\partial(g) - s(g) \in T(G)_g$ so (as $T(G) = G \times LG$) projects to a point in LG, which we call $\partial ln_{(G,s)}(g)$. We have defined the differential homomorphism $\partial ln_{(G,s)} : G \to LG$, which we write just as ∂ln_G, when s is understood or unique.
- $(G, s)^{\partial}$ is precisely $Ker(\partial ln_{(G,s)})$. This is a group of finite Morley rank definable in $\mathcal{U}$ and every group of finite Morley rank in DCF_0 arises this way (up to definable isomorphism).
- If ∂_s is the derivation of the "coordinate ring" corresponding to s then by considering its action on the local ring at the identity, ∂_s induces an additive differential map from LG to itself, in fact a connection on LG, which we call ∂_{LG} when s is understood or unique. $(LG)^{\partial}$ denotes the kernel of ∂_{LG}.

Algebraic D-groups and generalized logarithmic derivatives V

- Roughly speaking, if $x \in LG$ and $y = exp(x) \in G$ makes sense then $\partial ln_G(y) = \partial_{LG}(x)$. To be discussed by Daniel.
- If A is an abelian variety over K, and $\tilde{A}$ its universal extension by a vector group, then $L(\tilde{A})$ coincides with the dual of $H^1_{DR}(A)$, and $\partial_{L(\tilde{A})}$ coincides with the dual of the Gauss-Manin connection. To be discussed by Daniel.
- For G defined over the constants, and $s = 0$, ∂_{LG} is just $x \rightarrow \partial x$ on $\mathbb{G}_a^n$, so $(LG)^\partial = \mathcal{C}^n$.

Statement of Theorems I

The first has a rather strong differential algebraic geometric hypothesis: when we refer to LH, LG^{∂}, we refer to the points in $\mathcal{U}$, not just in K.

Theorem 1.2

Let $K = \mathbb{C}(t)^{alg}$ with $\partial = d/dt$. Let G be a commutative connected algebraic D-group defined over K, with no vectorial quotients (as an algebraic group). Let $x \in LG(K)$, and $y \in G(\mathcal{U})$ with $\partial ln_G(y) = \partial_{LG}(x)$. Assume
HX: *$x \notin LH + (LG)^{\partial}$ for any proper algebraic subgroup H of G defined over K.*
THEN $tr.deg(K(y)/K) = n = dim(G)$.

Statement of Theorems II

- The following version which we now find more attractive, has a weaker hypothesis $\mathbf{HX}_K$, closer in spirit to the arithmetic case, but we have to narrow the class of groups considered:
- Given a commutative algebraic group G over $K = \mathbb{C}(t)$, let A be its abelian part which we can write as $A_0 \cdot A_1$ where A_0 is defined over (descends to) $\mathbb{C}$ and A_1 has $\mathbb{C}$-trace 0.
- Let G^{sa} be the semiabelian part of G, that is the quotient of G by its maximal vector subgroup.
- Then let G_0^{sa} be the preimage of A_0 under the quotient map $G_1 \to A$.
- We call G_0^{sa} the semi-constant part of G.
- The hypothesis $(\mathbf{HG})_0$ is that G_0^{sa} is actually constant, namely isomorphic to a group defined over $\mathbb{C}$.

Statement of Theorems III

Theorem 1.3

Let again G be a commutative algebraic D-group defined over $K = \mathbb{C}(t)^{alg}$ and $\partial = d/dt$ on K. Let $x \in LG(K)$ and $y \in G(\mathcal{U})$ be such that $\partial ln_G(y) = \partial_{LG}(x)$. Assume $(\mathbf{HG})_0$, and $\mathbf{HX}_K$: $x \notin LH(K) + (LG)^{\partial}(K)$ for any proper algebraic subgroup H of G defined over K.
THEN $tr.deg(K(y)/K) = n = dim(G)$.

Ingredients of the proof include the semisimplicity of certain ∂-modules (Deligne), the Manin-Coleman-Chai Theorem of the kernel, as well as the "socle theorem" from model theory/differential algebraic geometry. We hope to sketch the proof in Lecture IV of the Bertrand-Pillay series.