Modnet tutorial at La Roche. IV. Outline of proof of functional Lindemann-Weierstrass

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Statement of Theorem

Theorem 1.1

Let $K = \mathbb{C}(t)^{alg}$ with $\partial = d/dt$. Let G be a commutative connected algebraic D-group defined over K, with no vectorial quotients (as an algebraic group). Let $x \in LG(K)$, and $y \in G(\mathcal{U})$ with $\partial ln_G(y) = \partial_{LG}(x)$. Assume

HX: *$x \notin LH + (LG)^{\partial}$ for any proper algebraic subgroup H of G defined over K.*

THEN $tr.deg(K(y)/K) = n = dim(G)$.

Theorem 1.2

Let again G be a commutative algebraic D-group defined over $K = \mathbb{C}(t)^{alg}$ with no vectorial quotients, and $\partial = d/dt$ on K. Let $x \in LG(K)$ and $y \in G(\mathcal{U})$ be such that $\partial ln_G(y) = \partial_{LG}(x)$. Assume $(\mathbf{HG})_0$, and

$\mathbf{HX}_K$: $x \notin LH(K) + (LG)^{\partial}(K)$ for any proper algebraic subgroup H of G defined over K.

THEN $tr.deg(K(y)/K) = n = dim(G)$.

Descent of $(HX)_K$

G is a commutative algebraic D-group over $K = \mathbb{C}(t)^{alg}$ with no vectorial quotients, and satisfying $(\mathbf{HG})_0$: the semiconstant part of G is constant.

Lemma 1.3

Suppose $x \in LG(K)$, and $x \notin LH(K) + (LG)^{\partial}(K)$ for any proper algebraic subgroup H of G defined over K. Let G_1 be a D-group quotient of G defined over K and x_1 the image of x in LG_1. Then $x_1 \notin LH_1 + (LG_1)^{\partial}$ for any proper algebraic subgroup H_1 of G_1 defined over K. Moreover G_1 also satisfies $(\mathbf{HG})_0$.

Proof. Uses results of Deligne that if A is an abelian variety over K and and $A = A_0 \cdot A_1$ is its decomposition into a group A_0 over $\mathbb{C}$ and a group A_1 with $\mathbb{C}$-trace 0, and $\tilde{A}$ is the universal extension of A by a vector group, then (a) $L\tilde{A}$ (with its canonical connection) is a semisimple ∂-module, and (b) $(L\tilde{A})^{\partial}(K) = L\tilde{A}_0(\mathbb{C})$.

Theorem of the kernel

As usual $K = \mathbb{C}(t)^{alg}$.
Let A be an abelian variety over K with $\mathbb{C}$-trace 0. Let $\tilde{A}$ be its universal extension of A by a vector group:
$0 \to W_A \to \tilde{A} \to A \to 0$.
So $\tilde{A}$ has a unique D-group structure. W_A, although not a D-subgroup of $\tilde{A}$ can be identified with a subspace of $L\tilde{A}$, and as such is its own Lie algebra. Let $V \subseteq W_A$ a D-subgroup of $\tilde{A}$ defined over K (necessarily different from W_A) and $G = \tilde{A}/V$ (also a D-group defined over K). The following is a version of Manin's theorem of the kernel, based on subsequent work of Coleman and Chai.

Lemma 1.4

Suppose $x \in LG(K)$, $y \in G(K)$, *and* $\partial ln_G(y) = \partial_{LG}(x)$. *Then* $x \in W_A/V$.

Proof. Daniel discussed it.

The socle theorem for DCF_0 I

Theorem 1.5

Suppose G is a connected commutative group of finite Morley rank definable in DCF_0, that is, definable in our saturated model $\mathcal{U}$ of DCF_0. Let Y be a definable subset of G of Morley degree 1, or better, an irreducible Kolchin closed subset of G. Let $S = Stab_G(Z)$. Suppose that S is finite. THEN Z is contained in a coset (translate) of H where H is the maximal connected definable subgroup of G which is "internal to the constants" (i.e. definably isomorphic to a group definable in $(\mathcal{C}, +, \cdot)$).

Note: there is a conjectural version for arbitrary groups of finite Morley rank, where "internal to the constants" is replaced by "internal to the family of nonmodular strongly minimal sets".

The socle theorem for DCF_0 II

Restatement (or consequence) for algebraic D-groups, with PHS's in place of groups:

Theorem 1.6

Let G be a commutative connected algebraic D-group, defined over an algebraically closed differential subfield K of $\mathcal{U}$. Let $Y \subseteq G$ be a coset (translate) of G^∂, and Z an irreducible differential algebraic subset of Y. Let $S < G^\partial$ be the stabilizer of Y (with respect to the regular action of G^∂ on Y). Suppose that S is finite. Then Z is contained in a coset (or translate or orbit) of H^∂ where H is the maximal connected isoconstant D-subgroup of G.

Proof of theorems I

- Let $dim(G) = n$.
- We fix x, and let $a = \partial_{LG}(x) \in LG(K)$.
- Let Y be the solution set of $\partial ln_G(-) = a$ (in $G = G(\mathcal{U})$ say).
- Y is coset (translate) of G^{∂} in G.
- As a "generic" point of Y over K has $tr.deg\ n$ over K (why ?), it suffices to prove that
  (*) Y has no proper differential algebraic subvariety, defined over K, i.e. that Y isolates a complete type over K,
- We prove this by induction on n.
- If $dim(G) = 1$ then $G = \mathbb{G}_m$ or an elliptic curve over $\mathbb{C}$ (why?), so Ax-Kirby does the job.

Proof of theorems II

- Now suppose $dim(G) = n > 1$. Suppose for a contradiction that (*) fails, witnessed by a proper irreducible differential algebraic subvariety Z of Y, defined over K.
- We are in the situation of Theorem 1.6. Let $S < G^{\partial}$ be the stabilizer of Z. S is a differential algebraic subgroup of G^{∂} so of the form $S' \cap G^{\partial}$ for some algebraic D-subgroup S' of G. Note that S, S' are defined over K, and are proper subgroups of G^{∂}, G, respectively.
- **Case I.** S, and so S', is infinite.
- Let $H = G/S'$, another connected algebraic D-group over K, of dimension positive but $< n$, and let $\pi : G \to H$ be the quotient homomorphism, which is also a homomorphism of algebraic D-groups.
- π induces $L\pi : LG \to LH$. Let $x' = (L\pi)(x) \in LH(K)$ and $a' = (L\pi)(a) \in LH(K)$.

Proof of theorems III

- Then $\partial_{LH}(x') = a'$ and $\pi(Y)$ is the solution space of $\partial ln_H(-) = a'$.
- The hypothesis $\mathbf{HX}$ clearly descends to the new data. Moreover assuming $(\mathbf{HG})_0$ as in Theorem 1.2, $(\mathbf{HX})_K$ is true of the new data by Lemma 1.3, and moreover the D-group H also satisfies $(\mathbf{HG})_0$.
- Hence in either the context of Theorem 1.1 or 1.2, we see, by induction hypothesis, that $\pi(Y)$ "isolates a complete type over K". In particular $\pi(Z) = \pi(Y)$, whereby $Y \subseteq Z + S'$, so $Y = Z + S$.
- But S stabilizes Z, so $Y = Z$, contradiction.

Proof of theorems IV

- **Case II.** S is finite.
- By Theorem 1.6, Z is contained in a translate, necessarily defined over K, of H^{∂}, where H is the maximal connected isoconstant D-subgroup of G.
- Let now $\pi : G \to G/H$ (a D-group) be the quotient map, and as in Case I, we obtain $\pi(x) = x' \in L(G/H)(K)$, $\pi(a) = a' \in L(G/H)(K)$, where $\partial_{L(G/H)}(x') = a'$, and $\pi(Y)$ is the solution set of $\partial ln_{G/H}(-) = a'$.
- But now $\pi(Z)$ is a K-rational point, say y' of $\pi(Y)$. Moreover the hypotheses $\mathbf{HX}$ and, where appropriate $(\mathbf{HX})_K$ and $(\mathbf{HG})_0$, are preserved.
- We now have three subcases:

Proof of theorems V

- **Subcase II(a)**. $H = G$. This says that G is isocontant, and so isomorphic over K to a group defined over $\mathbb{C}$ with trivial D-structure ($s = 0$). So we are in the constant case, where the conclusion of the Theorem is known (Ax-Kirby-Bertrand), giving us either the desired conclusion or a contradiction.
- **Subcase II(b)**. H is a nontrivial proper subgroup of G. But then $dim(G/H) < n$ is positive, and our K-rational point y' with $\partial ln_{G/H}(y') = \partial_{L(G/H)}(x')$ contradicts the induction hypothesis.
- **Subcase II(c)** $H = \{0\}$. So G has NO nontrivial isoconstant D-subgroups. One can conclude with a bit of work that G is of a special form: it's semiabelian part A is abelian and traceless.
- Namely, with notation as just before Lemma 1.4, there is an abelian variety A over K with $\mathbb{C}$-trace 0 such that G is a quotient of $\tilde{A}$ by a D-subgroup V of W_A.

Proof of theorems VI

- Note that Z is now a singleton $\{y\}$, with $y \in G(K)$, and $\partial ln_G(y) = \partial_{LG}(x)$.
- By Lemma 1.4, $x \in W_A/V$. But W_A/V is the Lie algebra of the algebraic subgroup W_A/V of G. This contradicts the hypothesis $(\mathbf{HX})_K$.
- The proof is complete.