Several motivic integrals

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Abstract

Notes after Immanuel Halupczok's tutorial at the La Roche-en-Ardenne MODNET meeting, April 2008.

We will compute some very basic motivic integrals, trying to give illustrations of both classical and Loeser-Cluckers formalisms. Sometimes we prefer to communicate intuition instead of being completely precise. All valued fields considered are in the Denef-Pas language (so we have valuation and the angular component maps). $K_0(...)$ stands for the Grothendieck ring under consideration, say $K_0(field)$ or $K_0(Psf)$.

Example 1 Let $C \subseteq \mathbb{A}^2$ be a curve, $C = V(f), f \in \mathbb{Z}[X,Y]$.

Consider formula $\phi =$ " $val(f(x,y)) > 0 \wedge x, y \in \mathbb{Z}_p$". Then for integral over it we have

$$\int_{\phi(\mathbb{Q}_p)} |dx||dy| = \#C(\mathbb{F}_p)\frac{1}{p^2}$$

$res(f(x,y)) = 0$,
$res(x), res(y) \in C(\mathbb{F}_p)$.
(If C is line $\rightsquigarrow \frac{1}{p}$).

$$\int_\phi |dx||dy| = [C]\frac{1}{\mathbb{L}^2} \subseteq K_0(...)\left[\frac{1}{\mathbb{L}}\right]$$

So it's only left to plug in the actual value of the $[C]\frac{1}{\mathbb{L}^2}$ class, which is $\#C(\mathbb{F}_p) \cdot \#\frac{1}{\mathbb{A}^2}(\mathbb{F}_p)$.

Example 2 Denote $squ(x) :=$ "$\exists z\ z^2 = x$" $\subseteq P_2$ and let us try to compute motivic measure of the following formula

$$\phi(x) := squ(x) \wedge val(x) = 0$$

Since anything we talk about is "for sufficientely large p's" we can safely assume $p \neq 2$. Note that

$$x \in \mathbb{Q}_p \text{ is } squ \Longleftrightarrow 2 \mid val(x) \wedge ac(x) \text{ is } squ$$

So we have (decomposing field as disjoint layers corresponding to decreasing valuation)

$$\int_{\phi(\mathbb{Q}_p)} | \, dx \, | = \frac{1}{p} \cdot \# squ_{\neq 0}(\mathbb{F}_p) = ...$$

$$\rightsquigarrow$$

$$(1 + p^{-2} + p^{-4} + \ldots) = \frac{1}{1 - p^{-2}}$$

$$\rightsquigarrow$$

$$\int_{\phi} | \, dx \, | = \frac{1}{\mathbb{L}} \cdot \frac{1}{1 - \mathbb{L}^{-2}} \cdot [squ_{\neq 0}] \in K_0(...) \left[\frac{1}{\mathbb{L}}\right] \cdot \left[\frac{1}{1 - \mathbb{L}^{-2}}\right]$$

Exercise Compute motivic measure of the [non-squares].

Example 3 Now we will treat a slightly more complicated case. Namely we find measure of the same formula, but over different field - $K((t))$.

Let $a = a_0 + a_1 t + a_2 t^2 + ...$

When $a \models \phi(x)$ holds? We can approximate set of squares from below with the following sequence (thus approximating measure of the resulting set as well, of course)

1^{st} *approximation (and its measure)*:

$$a_0 \models squ(x) \quad \rightsquigarrow \quad [squ] \cdot \frac{1}{\mathbb{L}}$$

2^{nd} *approximation*:

$$a_0 \models squ_{\neq 0} \ \vee \ a_0 = 0 \wedge a_1 \models squ \quad \rightsquigarrow \quad [squ_{\neq 0}] \cdot \frac{1}{\mathbb{L}} + \frac{1}{\mathbb{L}^2}$$

3^{rd} *approximation*:

$$a_0 \models squ_{\neq 0} \vee a_0 = 0 \wedge a_1 \models squ_{\neq 0} \vee a_0 = 0 \wedge a_1 = 0 \wedge a_2 \models squ \quad \rightsquigarrow \quad [squ_{\neq 0}] \cdot \frac{1}{\mathbb{L}} + [squ_{\neq 0}] \cdot \frac{1}{\mathbb{L}^2} + [squ] \cdot \frac{1}{\mathbb{L}^3}$$

$$\vdots$$

Continuing in the same way we see that essentially measure converges to

$$[squ_{\neq 0}] \cdot \left(\frac{1}{\mathbb{L}} + \frac{1}{\mathbb{L}^2} + \frac{1}{\mathbb{L}^3} + ... \right)$$

In order to make this answer precise (to be able to take limits) we need the completion of our Grothendieck ring $\widehat{K_0[\frac{1}{\mathbb{L}}]}$, where the answer finally sits.

Exercises And now for characters. Let

$\psi : \mathbb{Q}_p \to \mathbb{C}^\times,\ \psi(p\mathbb{Z}_p) = 1$

Compute the integrals (of different complexity):

$$\int_{\mathbb{Z}_p} \psi(x) \mid dx \mid = ?$$

$$\int_{p\mathbb{Z}_p} \psi(x) \mid dx \mid = ?$$

$$\int_{squ \,\cap\, \mathbb{Z}_p} \psi(x) \mid dx \mid = ?$$

$$\int_{\mathbb{Z}_p} \psi\left(\frac{1}{x}\right) \mid dx \mid = ?$$