MOTIVIC INTEGRATION

1 Finite counting and finite sums

Let $L = \{+, \times, 0, 1\}$ be the language of rings, and let φ be a formula in L in n free variables. For any integer N, we denote by $\varphi(\mathbb{Z}/N\mathbb{Z})$ the set points in $(\mathbb{Z}/N\mathbb{Z})^n$ satisfying the formula φ, and $\#\varphi(\mathbb{Z}/N\mathbb{Z})$ the number of such points.

***Example* 1.** *For instance, iff φ is the formula $\exists z,\ x = z^2 \wedge x \neq 0$, then $\#\varphi(\mathbb{Z}/N\mathbb{Z})$ is the number of non zero squares in $\mathbb{Z}/N\mathbb{Z}$*

. *If N is an odd prime number, then $\mathbb{Z}/N\mathbb{Z}^\times$ is a group, and $\#\varphi(\mathbb{Z}/N\mathbb{Z})$ is the cardinal of the image of the group morphism*

$$S : \mathbb{Z}/N\mathbb{Z}^\times \to \mathbb{Z}/N\mathbb{Z}^\times$$

$$x \mapsto x^2$$

As $\mathbb{Z}/N\mathbb{Z}$ is a field, $x^2 - 1 = (x+1)(x-1)$, and the kernel of S has cardinality two, so

$$\#\varphi(\mathbb{Z}/N\mathbb{Z}) = (N-1)/2$$

. *If N no longer a prime, it is the product of numbers $N_1, \dots N_p$ with N_i, N_j coprime when $i \neq j$, so, by the Chinese remainder theorem,*

$$\mathbb{Z}/N\mathbb{Z} \cong \mathbb{Z}/N_1\mathbb{Z} \times \cdots \times \mathbb{Z}/N_p\mathbb{Z}$$

and we get $\#\varphi(\mathbb{Z}/N\mathbb{Z}) = \#\varphi(\mathbb{Z}/N_1\mathbb{Z}) \cdot \#\varphi(\mathbb{Z}/N_2\mathbb{Z}) \cdots \#\varphi(\mathbb{Z}/N_p\mathbb{Z})$.

More generally, for group homomorphisms

$$\psi : (\mathbb{Z}/N\mathbb{Z})^+ \to \mathbb{C}^\times, \quad \chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$$

to the complex unit circle, and definable functions f and g in L, one can consider the sums $S_{\psi,\chi,N,\varphi,f,g}$ defined by

$$S_{\psi,\chi,N,\varphi,f,g} = \sum_{x \in \varphi(\mathbb{Z}/N\mathbb{Z})} \psi(f(x))\chi(g(x))$$

Note that these sums generalize Gauss sums $\sum_{x \in \mathbb{F}_p} \psi(x)\chi(x)$ over $\mathbb{F}_p$, obtained by taking N to be a prime, $\varphi(x)$ to be $x = x$, and $f = g = id$.

***Example* 2.** *Assume N is prime. Take for φ the formula $\exists z \neq 0,\ x = z^2$, and set $\psi(x) = exp(\frac{2\pi i x}{N})$, $\chi(x) = 1$ and $f = g = id$. Then*

$$S_\varphi = \sum_{x \in \varphi(\mathbb{Z}/N\mathbb{Z})} \psi(x) = \sum_{x \in \mathbb{F}_N^{2\times}} exp(\frac{2\pi i x}{N})$$

$$S_\varphi^2 = \sum_{x,y\in {\mathbb{F}_N^2}^\times} exp(\frac{2\pi i(x+y)}{N}) = 1/2 \sum_{x,y\in \mathbb{F}_N^\times} (\frac{x}{N})(\frac{y}{N}) exp(\frac{2\pi i(x+y)}{N})$$

where $(\frac{x}{N})$ *is the Legendre symbol defined by*

$$(\frac{x}{N}) = 1 \quad \textit{if } x \in \mathbb{F}_N^2$$

$$(\frac{x}{N}) = -1 \textit{ elsewhere}$$

Changing variable by putting $y = xz$*, we get* $(\frac{x}{N})(\frac{y}{N}) = (\frac{x}{N})^2(\frac{z}{N}) = (\frac{z}{N})$*, so*

$$S_\varphi^2 = 1/2 \sum_{x,z\in \mathbb{F}_N^\times} (\frac{z}{N}) exp(\frac{2\pi i x(1+z)}{N}) = 1/2 \sum_{z\in \mathbb{F}_N^\times} (\frac{z}{N}) \sum_{x\in \mathbb{F}_N^\times} exp(\frac{2\pi i x(1+z)}{N})$$

But $\sum_{x\in \mathbb{F}_N^\times} exp(\frac{2\pi i x(1+z)}{N}) = -1$ *if* $z \neq -1$*, and* $N-1$ *if* $z = -1$*, hence*

$$S_\varphi^2 = 1/2(\sum_{z\in \mathbb{F}_N^\times, z\neq -1} (\frac{z}{N}) + (N-1)(\frac{-1}{N}))$$

As there are as many squares as non squares in $\mathbb{F}_N^\times$*,* $\sum_{z\in \mathbb{F}_N^\times} (\frac{z}{N})$ *is zero, so*

$$S_\varphi^2 = \frac{N}{2}(\frac{-1}{N})$$

2 Integration

What if we consider infinite rings instead of $\mathbb{Z}/N\mathbb{Z}$?

***Definition* 3.** *Local fields of zero characteristic are finite extensions of* $\mathbb{Q}_p$*, the* $p-$*adic completion of* $\mathbb{Q}$ *for the norm* $|.|_p$*.*

***Proposition* 4.** $\mathbb{Q}_p$ *is the ring* $\{\sum_{i\geq l} a_i p^i : a_i \in \{0,\dots,p-1\}\}$*, with the ring operations coming from approximating any element by finite sums and calculating in the field* $\mathbb{Q}$*.*

Note that $\mathbb{Q}_p$ is a topological space with respect to the norm topology. It has the following properties.

***Properties* 5.**

1. $\mathbb{Z}_p$ *is a compact subring of* $\mathbb{Q}_p$*.*
2. $p^k\mathbb{Z}_p = \{x \in \mathbb{Q} : |x|_p \leq p^{-k}\}$ *are balls around* 0
3. $a + p^k\mathbb{Z}_p$ *and* $b + p^k\mathbb{Z}_p$ *are disjoint balls if* $a - b \notin p^k\mathbb{Z}_p$*.*

Recall that a topological group is a group which is a topological space, and whose multiplication and inverse maps are continuous. A topological space X is said to be locally compact if every point x in X has a compact neighbourhood. So $\mathbb{Q}_p$ is a locally compact topological group for the norm topology. We consider $\mathcal{B}$ the $\sigma-$algebra generated by open subsets of $\mathbb{Q}_p$, and construct a measure on $\mathcal{B}$. For calculus, and to be able to change variables, it would be convenient for this measure to be translation invariant.

Theorem 6. *Every locally compact topological group G has a real valued measure which is left (resp. right) invariant. This measure is unique up to multiplication, and is called the Haar measure of G.*

So we can construct a unique real valued translation invariant measure $|dx|$ on $\mathcal{B}$ by setting

$$|dx|(\mathbb{Z}_p) = 1$$

Let us now compute the measure of an open neighbourhood of 0, $p^k\mathbb{Z}_p$. For every k, we have

$$(p^k\mathbb{Z}_p) = \bigcup_{a_i \in \{0,\dots,p-1\}} ap^k + p^{k+1}\mathbb{Z}_p$$

As the union is disjoint, and $|dx|$ translation invariant and additive, we must have

$$|dx|(p^k\mathbb{Z}_p) = \sum_{a_i \in \{0,\dots,p-1\}} |dx|(ap^k + p^{k+1}\mathbb{Z}_p) = p.|dx|(p^{k+1}\mathbb{Z}_p)$$

so

$$|dx|(p^k\mathbb{Z}_p) = 1/p^k$$

***Corollary* 7.**

1. *There is a unique Haar measure $|dx|$ on $\mathbb{Q}_p$ with $|dx|(\mathbb{Z}_p) = 1$.*
2. *for all $a \in \mathbb{Q}_p$ and $k \in \mathbb{Z}$, $|dx|(a + p^k\mathbb{Z}_p) = 1/p^k$*
3. *Lebesgue theory applies, and one can integrate complex valued integrable functions.*

Now, let $L = \{+, \times, .^{-1}, |.|_p, 0, 1, =\}$ be the language of valued fields, and φ a formula in L in n free variables. Let us show that the set $\varphi(\mathbb{Q}_p)$ is measurable, by induction on the type of the formula. Note that if φ_0 is atomic, as equality is the only relation in the langage, there is a definable function $f : \mathbb{Q}_p^n \to \mathbb{Q}_p$, which is a term in the language L (that is, a composition of functions in L), so a continuous function, such that $\models \varphi_0(x) \leftrightarrow f(x) = 0$. So $\varphi_0(\mathbb{Q}_p) = f^{-1}(\{0\})$ is measurable. Note though that a definable function need not be continuous in general : $(x = 0 \wedge y = 0) \vee (x \neq 0 \wedge y = 1)$ is piecewise continuous. Now, if φ_1 is a boolean combination of atomic formulae, $\varphi_1(\mathbb{Q}_p)$ is a boolean combination of measurable sets, and is itself measurable. If φ_2 is of the form $\exists x \varphi_1$, where φ_1 is a boolean combination of atomic formulae, then $\varphi_2(\mathbb{Q}_p)$ is the projection of a measurable set, and is itself measurable. By induction on the type of formulae, $\varphi(\mathbb{Q}_p)$ is measurable for every formula φ, and one can integrate over $\varphi(\mathbb{Q}_p)$.

If $\psi : \mathbb{Q}_p^+ \to \mathbb{C}^\times$ is an additive character to the complex unit circle, trivial on $p\mathbb{Z}_p$, and, for all $a \in \mathbb{Z}_p$, $\chi_a : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ is a multiplicative character to the complex unit circle trivial on $1 + pa\mathbb{Z}_p$, for definable functions f and g in the language L, one can consider the integrals

$$I_{\psi,\chi,\mathbb{Q}_p,\varphi,f,g}(a) = \int_{x\in\varphi(\mathbb{Q}_p)} \psi(f(x))\chi_a(g(x))|dx|$$

if the function considered is absolutely integrable.

3 Motivic counting

Back to counting. We saw that the number of non zero squares in the field $\mathbb{F}_N$ is $(N-1)/2$. We would like to count them in a way that does not depend on the finite field we are in, and so that we could still spezialise it to counting solutions over finite fields. More generally, we will attach to every formula in the langage of ring, a counting measure.

We define the scissor group of ring formulas $\mathbb{S}_{cover}$ to be the free abelian group generated by ring formulas, with two families of relation :
Scissor relations : $[\varphi \vee \psi] = [\varphi] + [\psi] - [\varphi \wedge \psi]$
Congruence relations : $[\varphi] = n[\psi]$ if there exists a ring formula α such that for every pseudo-finite field K of characteristic zero, the interpretation of α gives a n to one correspondance between the tuples in K satisfying φ and the tupes in K satisfying ψ.

To enable a Fubini theorem for ring formulas, we introduce a product on $\mathbb{S}_{ring}$ by setting

$$[\varphi(x)][\psi(y)] = [\varphi(x) \wedge \psi(y)]$$

whenever the free variables occuring in x and y are distinct.

Let $\mathbb{K}$ be the free abelian group generated by varieties over $\mathbb{Q}$ with the relations :
Scissor relations : $[X] = [Z] + [X \setminus Z]$ if Z is a closed subvariety of X.
Congruence relations : $[X] = [Y]$ if X and Y are nonsingular projective varieties that give the same virtual Chow motive.

We call $\mathbb{K}$ the motivic scissor ring, note $\mathbb{L}$ for $[\mathbb{A}^1]$, the image of the affine line, and define $\mathbb{S}_{mot}$, the localized motivic scissor to be $\mathbb{K}[\mathbb{L}^{-1}] \otimes \mathbb{Q}$.

Theorem 8. *There exists a unique ring homorphism $\mathbb{S}_{cover} \to \mathbb{S}_{mot}$ satisfying the following property : if ϕ is a ring formula that is given by the conjunction of polynomial equations, then $[\phi]$ is sent to the affine variety defined by those polynomial equations.*

***Definition* 9.** *The map $\psi \mapsto [\psi] \in \mathbb{S}_{mot}$ will be called the motivic counting measure of the formula ψ.*

Theorem 10. *Let ψ be a ring formula and let $\sum a_i[X_i]$ be a representative of the motivic counting measure $[\psi]$ as a formal linear combination of varieties. For all r and for all but finetely many primes p, the number of solutions in $\mathbb{F}_{p^r}$ is*

$$\sum a_i \# X_i(\mathbb{F}_{p^r})$$

***Example* 11.** *If φ is the formula $\exists z,\ x = z^2 \wedge x \neq 0$, as the map $x \mapsto x^2$ gives a two-to-one correspondance between $x \neq 0$ and φ in every finite field, we have*

$$[\varphi] = (\mathbb{L} - 1)/2$$

4 Motivic integration

Ensuite calculs d'Immi (quelqu'un d'autre s'en charge)