Motivation: $p$-adic integration
From $\mathbb{Z}_p$ to $k[[t]]$

Geometric motivic integration

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Kontsevich invented motivic integration to strengthen the following result by Batyrev.

**Theorem (Batyrev)**

*If two complex Calabi-Yau varieties are birationally equivalent, then they have the same Betti numbers.*

Batyrev proved this result using $p$-adic integration and the Weil Conjectures. Kontsevich observed that Batyrev’s proof could be “geometrized”, avoiding the passage to finite fields and yielding a stronger result: **equality of Hodge numbers**. The key was to replace the $p$-adic integers by $\mathbb{C}[[t]]$, and $p$-adic integration by motivic integration.
Kontsevich presented these ideas at a famous “Lecture at Orsay” in 1995, but never published them. The theory was developed and generalized in the following directions:

- Denef and Loeser developed a theory of geometric motivic integration on arbitrary algebraic varieties over a field of characteristic zero. They also created a theory of arithmetic motivic integration, with good specialization properties to $p$-adic integrals in a general setting, using the model theory of pseudo-finite fields. The motivic integral appears here as a universal integral, specializing to the $p$-adic ones for almost all $p$. 
Loeser and Sebag constructed a theory of motivic integration on formal schemes and rigid varieties, working over an arbitrary complete discretely valued field with perfect residue field.

Cluckers and Loeser built a very general framework for motivic integration theories, based on model theory.

We will only discuss the so-called naïve geometric motivic integration on smooth algebraic varieties.
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We fix a prime number $p$. For any integer $n \geq 0$, we consider the compact group $\mathbb{Z}_p^n = (\mathbb{Z}_p)^n$ with its unique Haar measure $\mu$ such that $\mu(\mathbb{Z}_p^n) = 1$. 
Definition

We consider, for each $m, n \geq 0$, the natural projection

$$\pi_m : \mathbb{Z}_p^n \rightarrow (\mathbb{Z}_p/p^{m+1})^n$$

A cylinder $C$ in $\mathbb{Z}_p^n$ is a subset of the form $(\pi_m)^{-1}(C_m)$, for some $m \geq 0$ and some subset $C_m$ of $(\mathbb{Z}_p/p^{m+1})^n$. 
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Lemma

For any cylinder $C$, the series $(p^{-n(m+1)}|\pi_m(C)|)_{m \geq 0}$ is constant for $m \gg 0$, and its limit is equal to the Haar measure of $C$.

More precisely, if we choose $m_0 \geq 0$ such that $C = (\pi_{m_0})^{-1}(\pi_{m_0}(C))$, then for $m \geq m_0$

$$p^{-n(m+1)}|\pi_{m}(C)| = p^{-n(m_0+1)}|\pi_{m_0}(C)| = \mu(C)$$
Proof.

For $m \geq m_0$, $C$ can be written as a disjoint union

$$C = \bigsqcup_{a \in \pi_m(C)} (a + (p^{m+1}\mathbb{Z}_p)^n)$$

so by translation invariance of the Haar measure and the fact that $(p^{m+1}\mathbb{Z}_p)^n$ has measure $p^{-(m+1)n}$, we see that the measure of $C$ equals

$$p^{-(m+1)n}|\pi_m(C)|$$
If we identify $\mathbb{Z}_p$ with the ring of **Witt vectors** $W(\mathbb{F}_p)$, then the map $\pi_m$ simply corresponds to the truncation map

$$W(\mathbb{F}_p) \rightarrow W_{m+1}(\mathbb{F}_p) : (a_0, a_1, \ldots) \mapsto (a_0, a_1, \ldots, a_m)$$
The idea behind the theory of motivic integration is to make a similar construction, replacing $\mathcal{W}(\mathbb{F}_p)$ by the ring of formal power series $k[[t]]$ over some field $k$, and the map $\pi_m$ by the truncation map

$$k[[t]] \to k[t]/(t^{m+1}) : \sum_{i \geq 0} a_i t^i \mapsto \sum_{i=0}^{m} a_i t^i$$

The problem is to give meaning to the expression $|\pi_m(C)|$ if $C$ is a “cylinder” in $k[[t]]^n$ for infinite fields $k$, and to find a candidate to replace $p$ in the formula

$$\mu(C) = p^{-n(m+1)}|\pi_m(C)|$$
But interpreting the coefficients of a power series as affine coordinates, the set 

$$(k[[t]]/(t^{m+1}))^n$$

gets the structure of the set of $k$-points on an affine space $\mathbb{A}^{(m+1)n}_k$, and if we restrict to cylinders $C$ such that $\pi_m(C)$ is constructible in $\mathbb{A}^{(m+1)n}_k$, we can use the Grothendieck ring of varieties as a universal way to “count” points on constructible subsets of an algebraic variety.

The cardinality $p$ of $\mathbb{F}_p$ is replaced by the “number” of points on the affine line $\mathbb{A}^1_k$; this is the “Lefschetz motive” $\mathbb{L}$. 
The price to pay is that we leave classical integration theory since our value ring will be an abstract gadget (the Grothendieck ring of varieties) instead of $\mathbb{R}$. 
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The Grothendieck ring of varieties

\( k \) any field; \( k \)-variety = separated reduced \( k \)-scheme of finite type.

**Definition (Grothendieck ring of \( k \)-varieties)**

\( K_0(Var_k) \) abelian group
- generators: isomorphism classes \([X]\) of \( k \)-varieties \( X \)
- relations: \([X] = [X \setminus Y] + [Y]\) for \( Y \) closed in \( X \) ("scissor relations")

Ring multiplication: \([X_1] \cdot [X_2] = [(X_1 \times_k X_2)_{red}]\)

\( \mathbb{L} = [\mathbb{A}^1_k] \)
\( \mathcal{M}_k = K_0(Var_k)[\mathbb{L}^{-1}] \)

for \( X \) not reduced: \([X] := [X_{red}]\)
If $X$ is a $k$-variety and $C$ a **constructible subset** of $X$, then $C$ can be written as a disjoint union of locally closed subsets (subvarieties) of $X$ and this yields a well-defined class $[C]$ in $K_0(\text{Var}_k)$.

If a morphism of $k$-varieties $Y \to X$ is a Zariski-locally trivial fibration with fiber $F$, then $[Y] = [X] \cdot [F]$ in $K_0(\text{Var}_k)$. Indeed, using the scissor relations and Noetherian induction we can reduce to the case where the fibration is trivial.
By its very definition, the Grothendieck ring is a \textit{universal} additive and multiplicative invariant of \(k\)-varieties.

\[ \chi_{\text{top}} : K_0(\text{Var}_k) \rightarrow \mathbb{Z} \quad (\text{Euler characteristic}) \quad k = \mathbb{C} \]

\[ \# : K_0(\text{Var}_k) \rightarrow \mathbb{Z} \quad (\text{number of rational points}) \quad k \text{ finite} \]

So in a certain sense, taking the class \([X]\) of a \(k\)-variety \(X\) is the most general way to “count points” on \(X\).
Let $X$ be a variety over $k$. For each integer $n \geq 0$, we consider the functor

$$F_n : (k \text{-} alg) \rightarrow (Sets) : A \mapsto X(A[t]/(t^{n+1}))$$

**Proposition**

The functor $F_n$ is representable by a separated $k$-scheme of finite type $\mathcal{L}_n(X)$, called the $n$-th jet scheme of $X$. If $X$ is affine, then so is $\mathcal{L}_n(X)$. 

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Note: Saying that $\mathcal{L}_n(X)$ represents the functor $F_n$ simply means the following: for any $k$-algebra $A$, there exists a bijection

$$\phi_n^A : \mathcal{L}_n(X)(A) \to F_n(A) = X(A[t]/t^{n+1})$$

such that for any morphism of $k$-algebras $h : A \to B$, the square

$$\begin{array}{ccc}
\mathcal{L}_n(X)(A) & \xrightarrow{\phi_n^A} & X(A[t]/t^{n+1}) \\
\downarrow & & \downarrow \\
\mathcal{L}_n(X)(B) & \xrightarrow{\phi_n^B} & X(B[t]/t^{n+1})
\end{array}$$

commutes. This property uniquely determines the $k$-scheme $\mathcal{L}_n(X)$. 

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Idea of proof: We only consider the affine case

\[ X = \text{Spec } k[x_1, \ldots, x_r]/(f_1, \ldots, f_\ell) \]

i.e. \( X \) is the closed subvariety of \( \mathbb{A}_k^r \) defined by the equations \( f_1 = \ldots = f_\ell = 0 \).

Consider a tuple of variables \( (a_{1,0}, \ldots, a_{1,n}, a_{2,0}, \ldots, a_{r,n}) \) and the system of congruences

\[ f_j(\sum_{i=0}^{n} a_{1,i}t^i, \ldots, \sum_{i=0}^{n} a_{r,i}t^i) \equiv 0 \mod t^{n+1} \]

for \( j = 1, \ldots, \ell \).
If we replace the variables $a_{i,j}$ by elements of a $k$-algebra $A$, these congruences express exactly that the tuple

$$(a_{1,0} + \ldots + a_{1,n}t^n, \ldots, a_{r,0} + \ldots + a_{r,n}t^n) \in \mathbb{A}_k^r(A[t]/t^{n+1})$$

lies in $X(A[t]/(t^{n+1}))$. 
Developing, for each $j$,

\[ f_j\left(\sum_{i=0}^{n} a_{1,i}t^i, \ldots, \sum_{i=0}^{n} a_{r,i}t^i\right) \]

into a polynomial in $t$ and putting the coefficient of $t^i$ equal to 0 for $i = 0, \ldots, n$ yields a system of $\ell(n + 1)$ polynomial equations over $k$ in the variables $a_{i,j}$, and these define the scheme $\mathcal{L}_n(X)$ as a closed subscheme of $\mathbb{A}_k^{r(n+1)}$. 

\[ \square \]
Example: Let $X$ be the closed subvariety of $\mathbb{A}_k^2 = \text{Spec } k[x, y]$ defined by the equation $x^2 - y^3 = 0$. Then a point of $\mathcal{L}_2(X)$ with coordinates in some $k$-algebra $A$ is a couple

$$(x_0 + x_1 t + x_2 t^2, y_0 + y_1 t + y_2 t^2)$$

with $x_0, \ldots, y_2 \in A$ such that

$$(x_0 + x_1 t + x_2 t^2)^2 - (y_0 + y_1 t + y_2 t^2)^3 \equiv 0 \mod t^3$$
\[(x_0 + x_1 t + x_2 t^2)^2 - (y_0 + y_1 t + y_2 t^2)^3 \equiv 0 \pmod{t^3}\]

Working this out, we get the equations

\[
\begin{align*}
(x_0)^2 - (y_0)^3 &= 0 \\
2x_0x_1 - 3(y_0)^2y_1 &= 0 \\
(x_1)^2 + 2x_0x_2 - 3y_0(y_1)^2 - 3(y_0)^2y_2 &= 0
\end{align*}
\]

and if we view \(x_0, \ldots, y_2\) as affine coordinates, these equations define \(\mathcal{L}_2(X)\) as a closed subscheme of \(\mathbb{A}^6_k\).
For any $m \geq n$ and any $k$-algebra $A$, the truncation map

$$A[t]/t^{m+1} \to A[t]/t^{n+1}$$

defines a natural transformation of functors $F_m \to F_n$, so by Yoneda’s Lemma we get a natural truncation morphism of $k$-schemes

$$\pi^m_n : \mathcal{L}_m(X) \to \mathcal{L}_n(X)$$

This is the unique morphism such that for any $k$-algebra $A$, the square

$$\xymatrix{ X(A[t]/t^{m+1}) \ar[r] \ar[d]_{\phi^A_m} & X(A[t]/t^{n+1}) \ar[d]^ {\phi^A_n} \\
\mathcal{L}_m(X)(A) \ar[r]^-{\pi^m_n} & \mathcal{L}_n(X)(A) }$$

commutes.
Since the schemes $\mathcal{L}_n(X)$ are affine for affine $X$ and $\mathcal{L}_n(.)$ takes open covers to open covers, the morphisms $\pi^m_n$ are affine for any $k$-variety $X$, and we can take the projective limit

$$\mathcal{L}(X) = \lim_{\leftarrow n} \mathcal{L}_n(X)$$

in the category of $k$-schemes.
The scheme $\mathcal{L}(X)$ is called the arc scheme of $X$. It is not Noetherian, in general. It comes with natural projection morphisms

$$\pi_n : \mathcal{L}(X) \rightarrow \mathcal{L}_n(X)$$

For any field $F$ over $k$, we have a natural bijection

$$\mathcal{L}(X)(F) = X(F[[t]])$$

and the points of these sets are called $F$-valued arcs on $X$. The morphism $\pi_n$ maps an arc to its truncation modulo $t^{n+1}$. 
So by definition, a $F$-valued arc is a morphism

$$\psi : \text{Spec } F[[t]] \to X$$

i.e. a point of $X$ with coordinates in $F[[t]]$. The image of the closed point of $\text{Spec } F[[t]]$ is called the origin of the arc and denoted by $\psi(0)$; it is obtained by putting $t = 0$. An arc should be seen as an infinitesimal disc on $X$ with origin at $\psi(0)$.
We have $\mathcal{L}_0(X) = X$ and $\mathcal{L}_1(X)$ is the **tangent scheme** of $X$. The truncation morphism $\pi_0 : \mathcal{L}(X) \rightarrow X$ maps an arc $\psi$ to its origin $\psi(0)$. A morphism of $k$-varieties $h : Y \rightarrow X$ induces morphisms

\[
\mathcal{L}(h) : \mathcal{L}(Y) \rightarrow \mathcal{L}(X)
\]
\[
\mathcal{L}_n(h) : \mathcal{L}_n(Y) \rightarrow \mathcal{L}_n(X)
\]

commuting with the truncation maps.
If $X$ is smooth over $k$, of pure dimension $d$, then the morphisms $\pi^m_n$ are Zariski-locally trivial fibrations with fiber $\mathbb{A}^{d(m-n)}_k$ (use étale charts and the fact that $A[t]/(t^{m+1}) \to A[t]/(t^{n+1})$ is a nilpotent immersion for any $k$-algebra $A$).

Intuitively, arcs are local objects on $X$ and any smooth variety of pure dimension $d$ looks locally like an open subvariety of $\mathbb{A}^d_k$; but an element of $\mathcal{L}_n(\mathbb{A}^d_k)(A)$ is simply a $d$-tuple of elements in $A[t]/t^{n+1}$. 
If $X$ is singular, the spaces $\mathcal{L}_n(X)$ and $\mathcal{L}(X)$ are still quite mysterious. They contain a lot of information on the singularities of $X$. 
Example: Let us go back to our previous example, where $X \subset \mathbb{A}^2_k$ was given by the equation $x^2 - y^3 = 0$. For any $k$-algebra $A$, a point on $\mathcal{L}(X)$ with coordinates in $A$ is given by a couple

$$(x(t) = x_0 + x_1 t + x_2 t^2 + \ldots, y(t) = y_0 + y_1 t + y_2 t^2 + \ldots)$$

with $x_i, y_i \in A$, such that $x(t)^2 - y(t)^3 = 0$. Working this out yields an infinite number of polynomial equations in the variables $x_i, y_i$ and these realize $\mathcal{L}(X)$ as a closed subscheme of the infinite-dimensional affine space $\mathbb{A}^\infty_k = \text{Spec } k[x_0, y_0, x_1, y_1, \ldots]$. 
The truncation map

\[ \pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X) \]

sends \((x(t), y(t))\) to

\[ (x_0 + \ldots + x_n t^n, y_0 + \ldots + y_n t^n) \]

and (if \(A\) is a field) the origin of \((x(t), y(t))\) is simply the point \((x_0, y_0)\) on \(X\).
Note: If $k$ has characteristic zero, one can give an elegant construction of the schemes $\mathcal{L}_n(X)$ and $\mathcal{L}(X)$ using differential algebra. Assume that $X$ is affine, say given by polynomial equations

$$f_1(x_1, \ldots, x_r) = \ldots = f_\ell(x_1, \ldots, x_r)$$

in affine $r$-space $\mathbb{A}^r_k = \text{Spec } k[x_1, \ldots, x_r]$. Consider the $k$-algebra

$$B = k[y_{1,0}, \ldots, y_{r,0}, y_{1,1}, \ldots]$$

and the unique $k$-derivation $\delta : B \rightarrow B$ mapping $y_{i,j}$ to $y_{i,j+1}$ for each $i, j$. 
Then $\mathcal{L}(X)$ is isomorphic to the closed subscheme of $\text{Spec } B$ defined by the equations

$$\delta^{(i)}(f_q(y_{1,0}, \ldots, y_{r,0})) = 0$$

for $q = 1, \ldots, \ell$ and $i \in \mathbb{N}$. The point with coordinates $y_{i,j}$ corresponds to the arc

$$(\sum_{j \geq 0} \frac{y_{1,j}}{j!} t^j, \ldots, \sum_{j \geq 0} \frac{y_{r,j}}{j!} t^j)$$
Motivic integrals and change of variables

Copying the notion of cylinder and the description of its Haar measure, we can define a **motivic measure** on a class of subsets of the arc space $\mathcal{L}(X)$. From now on, we assume that $X$ is smooth over $k$, of pure dimension $d$. 
**Definition**

A **cylinder** in $\mathcal{L}(X)$ is a subset $C$ of the form $(\pi_m)^{-1}(C_m)$, with $m \geq 0$ and $C_m$ a constructible subset of $\mathcal{L}_m(X)$.

Note that the set of cylinders in $\mathcal{L}(X)$ is a Boolean algebra, i.e. it is closed under complements, finite unions and finite intersections.
Definition-Lemma

Let $C$ be a cylinder in $\mathcal{L}(X)$, and choose $m \geq 0$ such that $C = (\pi_m)^{-1}(C_m)$ with $C_m$ constructible in $\mathcal{L}_m(X)$. The value

$$\mu(C) := [\pi_m(C)]\mathbb{L}^{-d(m+1)} \in \mathcal{M}_k$$

does not depend on $m$, and is called the motivic measure $\mu(C)$ of $C$.

This follows immediately from the fact that the truncation morphisms $\pi^n_m$ are Zariski-locally trivial fibrations with fiber $\mathbb{A}^{d(n-m)}_k$. 
Example

If \( C = \mathcal{L}(X) \), then \( \mu(C) = \mathbb{L}^{-d}[X] \).

The normalization factor \( \mathbb{L}^{-d} \) is added in accordance with the \( p \)-adic case, where the ring of integers gets measure one (rather then the cardinality of the residue field).

Note that the motivic measure \( \mu \) is additive w.r.t. finite disjoint unions.
In the general theory of motivic integration, one constructs a much larger class of measurable sets and one defines the motivic measure via approximation by cylinders. This necessitates replacing $\mathcal{M}_k$ by a certain “dimensional completion”. We will not consider this generalization here.
**Definition**

We say that a function

\[ \alpha : \mathcal{L}(X) \rightarrow \mathbb{N} \cup \{\infty\} \]

is **integrable** if its image is finite, and if \( \alpha^{-1}(i) \) is a cylinder for each \( i \in \mathbb{N} \).

We define the **motivic integral** of \( \alpha \) by

\[ \int_{\mathcal{L}(X)} \mathbb{L}^{-\alpha} = \sum_{i \in \mathbb{N}} \mu(\alpha^{-1}(i)) \mathbb{L}^{-i} \in \mathcal{M}_k \]
The central and most powerful tool in the theory of motivic integration is the change of variables formula. For its precise statement, we need some auxiliary notation. For any $k$-variety $Y$, any ideal sheaf $\mathcal{J}$ on $Y$ and any arc

$$\psi : \text{Spec } F[[t]] \to Y$$

on $Y$, we define the order of $\mathcal{J}$ at $\psi$ by

$$\text{ord}_t \mathcal{J}(\psi) = \min\{\text{ord}_t \psi^* f \mid f \in \mathcal{J}_{\psi}(0)\}$$

where $\text{ord}_t$ is the $t$-adic valuation. In this way, we obtain a function

$$\text{ord}_t \mathcal{J} : \mathcal{L}(Y) \to \mathbb{N} \cup \{\infty\}$$

whose fibers over $\mathbb{N}$ are cylinders.
Let $h : Y \to X$ be a proper birational morphism, with $Y$ smooth over $k$, and denote by $\text{Jac}_h$ the Jacobian ideal of $h$. Let $\alpha$ be an integrable function on $\mathcal{L}(X)$, and assume that $\text{ord}_t \text{Jac}_h$ takes only finitely many values on each fiber of $\alpha \circ \mathcal{L}(h)$ over $\mathbb{N}$. Then

$$\int_{\mathcal{L}(X)} -\alpha = \int_{\mathcal{L}(Y)} -((\alpha \circ \mathcal{L}(h)) + \text{ord}_t \text{Jac}_h)$$

in $\mathcal{M}_k$. 

Theorem (Change of variables formula)
The very basic idea behind the change of variables formula is the following: if we denote by $V$ the closed subscheme of $Y$ defined by the Jacobian ideal $Jac_h$, and by $U$ its image under $h$, then the morphism $h : Y - V \to X - U$ is an isomorphism. Combined with the properness of $h$, this implies that

$$\mathcal{L}(h) : \mathcal{L}(Y) - \mathcal{L}(V) \to \mathcal{L}(X) - \mathcal{L}(U)$$

is a bijection; but $\mathcal{L}(V)$ and $\mathcal{L}(Y)$ have measure zero in $\mathcal{L}(Y)$, resp. $\mathcal{L}(X)$ (w.r.t. a certain more refined motivic measure) so it is reasonable to expect that there exists a change of variables formula.
The jet spaces $\mathcal{L}_n(Y)$, however, are “contracted” under the morphism

$$\mathcal{L}_n(h) : \mathcal{L}_n(Y) \to \mathcal{L}_n(X)$$

and this affects the motivic measure of cylinders. The “contraction factor” is measured by the Jacobian.