Geometric motivic integration

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Introduction

Kontsevich invented motivic integration to strengthen the following result by Batyrev.

Theorem (Batyrev)

If two complex Calabi-Yau varieties are birationally equivalent, then they have the same Betti numbers.

Batyrev proved this result using p-adic integration and the Weil Conjectures. Kontsevich observed that Batyrev's proof could be "geometrized", avoiding the passage to finite fields and yielding a stronger result: equality of Hodge numbers. The key was to replace the p-adic integers by $\mathbb{C}[[t]]$, and p-adic integration by motivic integration.

Kontsevich presented these ideas at a famous "Lecture at Orsay" in 1995, but never published them. The theory was developed and generalized in the following directions:

- Denef and Loeser developed a theory of geometric motivic integration on arbitrary algebraic varieties over a field of characteristic zero.
  They also created a theory of arithmetic motivic integration, with good specialization properties to p-adic integrals in a general setting, using the model theory of pseudo-finite fields. The motivic integral appears here as a universal integral, specializing to the p-adic ones for almost all p.

- Loeser and Sebag constructed a theory of motivic integration on formal schemes and rigid varieties, working over an arbitrary complete discretely valued field with perfect residue field.
- Cluckers and Loeser built a very general framework for motivic integration theories, based on model theory.

We will only discuss the so-called naïve geometric motivic integration on smooth algebraic varieties.

Outline

1. Motivation: p-adic integration

2. From $\mathbb{Z}_p$ to $k[[t]]$
    - The Grothendieck ring of varieties
    - Arc spaces
    - Motivic integrals and change of variables

Basic references:

- J. Denef & F. Loeser, "Geometry on arc spaces of algebraic varieties", European Congress of Mathematics, Vol. 1 (Barcelona, 2000), Progr. Math. 201, 2001
- W. Veys, "Arc spaces, motivic integration and stringy invariants", Advanced Studies in Pure Mathematics 43 , Proc. of "Singularity Theory and its applications, Sapporo, 16-25 september 2003" (2006)

p-adic integration

We fix a prime number p. For any integer $n \geq 0$, we consider the compact group $\mathbb{Z}_p^n = (\mathbb{Z}_p)^n$ with its unique Haar measure μ such that $\mu(\mathbb{Z}_p^n) = 1$.

Definition

We consider, for each $m, n \geq 0$, the natural projection

$$\pi_m : \mathbb{Z}_p^n \to (\mathbb{Z}_p/p^{m+1})^n$$

A cylinder C in $\mathbb{Z}_p^n$ is a subset of the form $(\pi_m)^{-1}(C_m)$, for some $m \geq 0$ and some subset C_m of $(\mathbb{Z}_p/p^{m+1})^n$.

Lemma

For any cylinder C, the series $(p^{-n(m+1)}|\pi_m(C)|)_{m\geq 0}$ is constant for $m \gg 0$, and its limit is equal to the Haar measure of C. More precisely, if we choose $m_0 \geq 0$ such that $C = (\pi_{m_0})^{-1}(\pi_{m_0}(C))$, then for $m \geq m_0$

$$p^{-n(m+1)}|\pi_m(C)| = p^{-n(m_0+1)}|\pi_{m_0}(C)| = \mu(C)$$

Proof.

For $m \geq m_0$, C can be written as a disjoint union

$$C = \bigsqcup_{a \in \pi_m(C)} (a + (p^{m+1}\mathbb{Z}_p)^n)$$

so by translation invariance of the Haar measure and the fact that $(p^{m+1}\mathbb{Z}_p)^n$ has measure $p^{-(m+1)n}$, we see that the measure of C equals

$$p^{-(m+1)n}|\pi_m(C)|$$

□

From $\mathbb{Z}_p$ to $k[[t]]$

If we identify $\mathbb{Z}_p$ with the ring of Witt vectors $W(\mathbb{F}_p)$, then the map π_m simply corresponds to the truncation map

$$W(\mathbb{F}_p) \to W_{m+1}(\mathbb{F}_p) : (a_0, a_1, \ldots) \mapsto (a_0, a_1, \ldots, a_m)$$

The idea behind the theory of motivic integration is to make a similar construction, replacing $W(\mathbb{F}_p)$ by the ring of formal power series $k[[t]]$ over some field k, and the map π_m by the truncation map

$$k[[t]] \to k[t]/(t^{m+1}) : \sum_{i\geq 0} a_i t^i \mapsto \sum_{i=0}^{m} a_i t^i$$

The problem is to give meaning to the expression $|\pi_m(C)|$ if C is a "cylinder" in $k[[t]]^n$ for infinite fields k, and to find a candidate to replace p in the formula

$$\mu(C) = p^{-n(m+1)}|\pi_m(C)|)$$

But interpreting the coefficients of a power series as affine coordinates, the set

$$(k[[t]]/(t^{m+1}))^n$$

gets the structure of the set of k-points on an affine space $\mathbb{A}_k^{(m+1)n}$, and if we restrict to cylinders C such that $\pi_m(C)$ is constructible in $\mathbb{A}_k^{(m+1)n}$, we can use the Grothendieck ring of varieties as a universal way to "count" points on constructible subsets of an algebraic variety.

The cardinality p of $\mathbb{F}_p$ is replaced by the "number" of points on the affine line $\mathbb{A}_k^1$; this is the "Lefschetz motive" $\mathbb{L}$.

The price to pay is that we leave classical integration theory since our value ring will be an abstract gadget (the Grothendieck ring of varieties) instead of $\mathbb{R}$.

The Grothendieck ring of varieties

k any field; k-variety= separated reduced k-scheme of finite type.

Definition (Grothendieck ring of k-varieties)

$K_0(Var_k)$ abelian group

- generators: isomorphism classes $[X]$ of k-varieties X
- relations: $[X] = [X \setminus Y] + [Y]$ for Y closed in X ("scissor relations")

Ring multiplication: $[X_1] \cdot [X_2] = [(X_1 \times_k X_2)_{red}]$

$\mathbb{L} = [\mathbb{A}^1_k] \qquad \mathcal{M}_k = K_0(Var_k)[\mathbb{L}^{-1}]$

for X not reduced: $[X] := [X_{red}]$

If X is a k-variety and C a constructible subset of X, then C can be written as a disjoint union of locally closed subsets (subvarieties) of X and this yields a well-defined class $[C]$ in $K_0(Var_k)$.

If a morphism of k-varieties $Y \to X$ is a Zariski-locally trivial fibration with fiber F, then $[Y] = [X] \cdot [F]$ in $K_0(Var_k)$. Indeed, using the scissor relations and Noetherian induction we can reduce to the case where the fibration is trivial.

By its very definition, the Grothendieck ring is a universal additive and multiplicative invariant of k-varieties.

$\rightsquigarrow$ specialization morphisms of rings

$$\chi_{top} \quad : \quad K_0(Var_k) \to \mathbb{Z} \quad \text{(Euler characteristic)} \qquad k = \mathbb{C}$$

$$\sharp \quad : \quad K_0(Var_k) \to \mathbb{Z} \quad \text{(number of rational points)} \quad k \text{ finite}$$

So in a certain sense, taking the class $[X]$ of a k-variety X is the most general way to "count points" on X.

Arc spaces

Let X be a variety over k. For each integer $n \geq 0$, we consider the functor

$$F_n : (k - alg) \to (Sets) : A \mapsto X(A[t]/(t^{n+1}))$$

Proposition

The functor F_n is representable by a separated k-scheme of finite type $\mathcal{L}_n(X)$, called the n-th jet scheme of X. If X is affine, then so is $\mathcal{L}_n(X)$.

Note: Saying that $\mathcal{L}_n(X)$ represents the functor F_n simply means the following: for any k-algebra A, there exists a bijection

$$\phi_n^A : \mathcal{L}_n(X)(A) \to F_n(A) = X(A[t]/t^{n+1})$$

such that for any morphism of k-algebras $h : A \to B$, the square

$$\begin{array}{ccc} \mathcal{L}_n(X)(A) & \xrightarrow{\phi_n^A} & X(A[t]/t^{n+1}) \\ \downarrow & & \downarrow \\ \mathcal{L}_n(X)(B) & \xrightarrow{\phi_n^B} & X(B[t]/t^{n+1}) \end{array}$$

commutes. This property uniquely determines the k-scheme $\mathcal{L}_n(X)$.

Idea of proof: We only consider the affine case

$$X = \mathrm{Spec}\, k[x_1, \ldots, x_r]/(f_1, \ldots, f_\ell)$$

i.e. X is the closed subvariety of $\mathbb{A}^r_k$ defined by the equations $f_1 = \ldots = f_\ell = 0$.
Consider a tuple of variables $(a_{1,0}, \ldots, a_{1,n}, a_{2,0}, \ldots, a_{r,n})$ and the system of congruences

$$f_j(\sum_{i=0}^{n} a_{1,i}t^i, \ldots, \sum_{i=0}^{n} a_{r,i}t^i) \equiv 0 \mod t^{n+1}$$

for $j = 1, \ldots, \ell$.

If we replace the variables $a_{i,j}$ by elements of a k-algebra A, these congruences express exactly that the tuple

$$(a_{1,0} + \ldots + a_{1,n}t^n, \ldots, a_{r,0} + \ldots + a_{r,n}t^n) \in \mathbb{A}^r_k(A[t]/t^{n+1})$$

lies in $X(A[t]/(t^{n+1}))$.

Developing, for each j,

$$f_j(\sum_{i=0}^{n} a_{1,i}t^i, \ldots, \sum_{i=0}^{n} a_{r,i}t^i)$$

into a polynomial in t and putting the coefficient of t^i equal to 0 for $i = 0, \ldots, n$ yields a system of $\ell(n+1)$ polynomial equations over k in the variables $a_{i,j}$, and these define the scheme $\mathcal{L}_n(X)$ as a closed subscheme of $\mathbb{A}_k^{r(n+1)}$. □

Example: Let X be the closed subvariety of $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$ defined by the equation $x^2 - y^3 = 0$. Then a point of $\mathcal{L}_2(X)$ with coordinates in some k-algebra A is a couple

$$(x_0 + x_1 t + x_2 t^2, y_0 + y_1 t + y_2 t^2)$$

with $x_0, \ldots, y_2 \in A$ such that

$$(x_0 + x_1 t + x_2 t^2)^2 - (y_0 + y_1 t + y_2 t^2)^3 \equiv 0 \quad \mod t^3$$

$$(x_0 + x_1 t + x_2 t^2)^2 - (y_0 + y_1 t + y_2 t^2)^3 \equiv 0 \mod t^3$$

Working this out, we get the equations

$$\begin{cases} (x_0)^2 - (y_0)^3 & = & 0 \\ 2x_0x_1 - 3(y_0)^2 y_1 & = & 0 \\ (x_1)^2 + 2x_0x_2 - 3y_0(y_1)^2 - 3(y_0)^2 y_2 & = & 0 \end{cases}$$

and if we view $x_0, \ldots, y_2$ as affine coordinates, these equations define $\mathcal{L}_2(X)$ as a closed subscheme of $\mathbb{A}^6_k$. □

For any $m \geq n$ and any k-algebra A, the truncation map

$$A[t]/t^{m+1} \to A[t]/t^{n+1}$$

defines a natural transformation of functors $F_m \to F_n$, so by Yoneda's Lemma we get a natural truncation morphism of k-schemes

$$\pi_n^m : \mathcal{L}_m(X) \to \mathcal{L}_n(X)$$

This is the unique morphism such that for any k-algebra A, the square

$$\begin{array}{ccc} X(A[t]/t^{m+1}) & \longrightarrow & X(A[t]/t^{n+1}) \\ \phi_m^A \downarrow & & \downarrow \phi_n^A \\ \mathcal{L}_m(X)(A) & \xrightarrow{\pi_n^m} & \mathcal{L}_n(X)(A) \end{array}$$

commutes.

Since the schemes $\mathcal{L}_n(X)$ are affine for affine X and $\mathcal{L}_n(.)$ takes open covers to open covers, the morphisms π_n^m are affine for any k-variety X, and we can take the projective limit

$$\mathcal{L}(X) = \lim_{\longleftarrow n} \mathcal{L}_n(X)$$

in the category of k-schemes.

The scheme $\mathcal{L}(X)$ is called the arc scheme of X. It is not Noetherian, in general. It comes with natural projection morphisms

$$\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X)$$

For any field F over k, we have a natural bijection

$$\mathcal{L}(X)(F) = X(F[[t]])$$

and the points of these sets are called F-valued arcs on X. The morphism π_n maps an arc to its truncation modulo t^{n+1}.

So by definition, a F-valued arc is a morphism

$$\psi : \operatorname{Spec} F[[t]] \to X$$

i.e. a point of X with coordinates in $F[[t]]$. The image of the closed point of $\operatorname{Spec} F[[t]]$ is called the origin of the arc and denoted by $\psi(0)$; it is obtained by putting $t = 0$. An arc should be seen as an infinitesimal disc on X with origin at $\psi(0)$.

We have $\mathcal{L}_0(X) = X$ and $\mathcal{L}_1(X)$ is the tangent scheme of X. The truncation morphism $\pi_0 : \mathcal{L}(X) \to X$ maps an arc ψ to its origin $\psi(0)$. A morphism of k-varieties $h : Y \to X$ induces morphisms

$$\begin{aligned} \mathcal{L}(h) &: \mathcal{L}(Y) \to \mathcal{L}(X) \\ \mathcal{L}_n(h) &: \mathcal{L}_n(Y) \to \mathcal{L}_n(X) \end{aligned}$$

commuting with the truncation maps.

If X is smooth over k, of pure dimension d, then the morphisms π_n^m are Zariski-locally trivial fibrations with fiber $\mathbb{A}_k^{d(m-n)}$ (use étale charts and the fact that $A[t]/(t^{m+1}) \to A[t]/(t^{n+1})$ is a nilpotent immersion for any k-algebra A).
Intuitively, arcs are local objects on X and any smooth variety of pure dimension d looks locally like an open subvariety of $\mathbb{A}_k^d$; but an element of $\mathcal{L}_n(\mathbb{A}_k^d)(A)$ is simply a d-tuple of elements in $A[t]/t^{n+1}$.

If X is singular, the spaces $\mathcal{L}_n(X)$ and $\mathcal{L}(X)$ are still quite mysterious. They contain a lot of information on the singularities of X.

Example: Let us go back to our previous example, where $X \subset \mathbb{A}^2_k$ was given by the equation $x^2 - y^3 = 0$. For any k-algebra A, a point on $\mathcal{L}(X)$ with coordinates in A is given by a couple

$$(x(t) = x_0 + x_1 t + x_2 t^2 + \ldots, y(t) = y_0 + y_1 t + y_2 t^2 + \ldots)$$

with x_i, $y_i \in A$, such that $x(t)^2 - y(t)^3 = 0$. Working this out yields an infinite number of polynomial equations in the variables x_i, y_i and these realize $\mathcal{L}(X)$ as a closed subscheme of the infinite-dimensional affine space $\mathbb{A}^\infty_k = \mathrm{Spec}\, k[x_0, y_0, x_1, y_1, \ldots]$.

The truncation map

$$\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X)$$

sends $(x(t), y(t))$ to

$$(x_0 + \ldots + x_n t^n, y_0 + \ldots + y_n t^n)$$

and (if A is a field) the origin of $(x(t), y(t))$ is simply the point (x_0, y_0) on X. $\square$

Note: If k has characteristic zero, one can give an elegant construction of the schemes $\mathcal{L}_n(X)$ and $\mathcal{L}(X)$ using differential algebra. Assume that X is affine, say given by polynomial equations

$$f_1(x_1,\ldots,x_r)=\ldots=f_\ell(x_1,\ldots,x_r)$$

in affine r-space $\mathbb{A}^r_k=\mathrm{Spec}\,k[x_1,\ldots,x_r]$. Consider the k-algebra

$$B=k[y_{1,0},\ldots,y_{r,0},y_{1,1},\ldots]$$

and the unique k-derivation $\delta:B\to B$ mapping $y_{i,j}$ to $y_{i,j+1}$ for each i,j.

Then $\mathcal{L}(X)$ is isomorphic to the closed subscheme of $\mathrm{Spec}\, B$ defined by the equations

$$\delta^{(i)}(f_q(y_{1,0},\ldots,y_{r,0}))=0$$

for $q=1,\ldots,\ell$ and $i\in\mathbb{N}$. The point with coordinates $y_{i,j}$ corresponds to the arc

$$(\sum_{j\geq 0}\frac{y_{1,j}}{j!}t^j,\ldots,\sum_{j\geq 0}\frac{y_{r,j}}{j!}t^j)$$

Motivic integrals and change of variables

Copying the notion of cylinder and the description of its Haar measure, we can define a motivic measure on a class of subsets of the arc space $\mathcal{L}(X)$. From now on, we assume that X is smooth over k, of pure dimension d.

Definition

A cylinder in $\mathcal{L}(X)$ is a subset C of the form $(\pi_m)^{-1}(C_m)$, with $m \geq 0$ and C_m a constructible subset of $\mathcal{L}_m(X)$.

Note that the set of cylinders in $\mathcal{L}(X)$ is a Boolean algebra, i.e. it is closed under complements, finite unions and finite intersections.

Definition-Lemma

Let C be a cylinder in $\mathcal{L}(X)$, and choose $m \geq 0$ such that $C = (\pi_m)^{-1}(C_m)$ with C_m constructible in $\mathcal{L}_m(X)$. The value

$$\mu(C) := [\pi_m(C)]\mathbb{L}^{-d(m+1)} \quad \in \mathcal{M}_k$$

does not depend on m, and is called the motivic measure $\mu(C)$ of C.

This follows immediately from the fact that the truncation morphisms π^n_m are Zariski-locally trivial fibrations with fiber $\mathbb{A}_k^{d(n-m)}$.

Example

If $C = \mathcal{L}(X)$, then $\mu(C) = \mathbb{L}^{-d}[X]$.

The normalization factor $\mathbb{L}^{-d}$ is added in accordance with the p-adic case, where the ring of integers gets measure one (rather then the cardinality of the residue field).
Note that the motivic measure μ is additive w.r.t. finite disjoint unions.

In the general theory of motivic integration, one constructs a much larger class of measurable sets and one defines the motivic measure via approximation by cylinders. This necessitates replacing $\mathcal{M}_k$ by a certain "dimensional completion". We will not consider this generalization here.

Definition

We say that a function

$$\alpha : \mathcal{L}(X) \to \mathbb{N} \cup \{\infty\}$$

is integrable if its image is finite, and if $\alpha^{-1}(i)$ is a cylinder for each $i \in \mathbb{N}$.

We define the motivic integral of α by

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-\alpha} = \sum_{i \in \mathbb{N}} \mu(\alpha^{-1}(i))\mathbb{L}^{-i} \quad \in \mathcal{M}_k$$

The central and most powerful tool in the theory of motivic integration is the change of variables formula. For its precise statement, we need some auxiliary notation. For any k-variety Y, any ideal sheaf $\mathcal{J}$ on Y and any arc

$$\psi : \operatorname{Spec} F[[t]] \to Y$$

on Y, we define the order of $\mathcal{J}$ at ψ by

$$ord_t \mathcal{J}(\psi) = \min\{ord_t \psi^* f \mid f \in \mathcal{J}_{\psi(0)}\}$$

where ord_t is the t-adic valuation. In this way, we obtain a function

$$ord_t \mathcal{J} : \mathcal{L}(Y) \to \mathbb{N} \cup \{\infty\}$$

whose fibers over $\mathbb{N}$ are cylinders.

Theorem (Change of variables formula)

Let $h : Y \to X$ be a proper birational morphism, with Y smooth over k, and denote by Jac_h the Jacobian ideal of h. Let α be an integrable function on $\mathcal{L}(X)$, and assume that $ord_t Jac_h$ takes only finitely many values on each fiber of $\alpha \circ \mathcal{L}(h)$ over $\mathbb{N}$. Then

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-\alpha} = \int_{\mathcal{L}(Y)} \mathbb{L}^{-((\alpha \circ \mathcal{L}(h)) + ord_t Jac_h)}$$

in $\mathcal{M}_k$.

The very basic idea behind the change of variables formula is the following: if we denote by V the closed subscheme of Y defined by the Jacobian ideal Jac_h, and by U its image under h, then the morphism $h : Y - V \to X - U$ is an isomorphism. Combined with the properness of h, this implies that

$$\mathcal{L}(h) : \mathcal{L}(Y) - \mathcal{L}(V) \to \mathcal{L}(X) - \mathcal{L}(U)$$

is a bijection; but $\mathcal{L}(V)$ and $\mathcal{L}(Y)$ have measure zero in $\mathcal{L}(Y)$, resp. $\mathcal{L}(X)$ (w.r.t. a certain more refined motivic measure) so it is reasonable to expect that there exists a change of variables formula.

The jet spaces $\mathcal{L}_n(Y)$, however, are "contracted" under the morphism

$$\mathcal{L}_n(h) : \mathcal{L}_n(Y) \to \mathcal{L}_n(X)$$

and this affects the motivic measure of cylinders. The "contraction factor" is measured by the Jacobian.