Combinatorial geometries of the field extensions

Jakub Gismatullin

Mathematisches Institut der Universität Freiburg
Instytut Matematyczny Uniwersytetu Wrocławskiego

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$x \sim y \Leftrightarrow \mathrm{cl}(\{x\}) = \mathrm{cl}(\{y\}) \quad \mathrm{cl}_\sim(\frac{A}{\sim}) = \frac{\mathrm{cl}(A) \setminus \mathrm{cl}(\emptyset)}{\sim}$

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Methods: 'rank one' and 'higher rank' group configuration of Hrushovski (for $G_a(L) \rtimes G_m(L)$)

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apply this to $K \subseteq L \subseteq \widehat{L}$

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$$\{\mathrm{acl}_K(x^a * y^b * z^c) \colon a, b, c \in \mathbb{Z}\} \cong \mathbb{P}^3(\mathbb{Q})$$

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Evans and Hrushovski proved, that any projective plane in $\mathbb{G}(L/K)$ is a subplane of a plane like above.

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- $\widehat{K} \cap L$ *or* $\mathbb{Q}(\sqrt{(-d)})$ *for some* $d < \omega$