The Lindemann-Weierstrass theorem for semi-abelian varieties over function fields

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Introduction

The tutorial was about a functional version of the Lindemann-Weierstrass theorem, which is a special case of the Schanuel conjecture:

Conjecture 1. *(Schanuel) Let $z_1, \dots, z_n$ be n complex numbers linearly independent over $\mathbf{Q}$. Then the following inequality holds:*

$$\mathrm{tr.deg}_{\mathbf{Q}} \mathbf{Q}(z_1, \dots, z_n, e^{z_1}, \dots, e^{z_n}) \geqslant n$$

Recall that the general statement is not proved, but the result is known if the z_i's are algebraic:

Theorem 0.1. *(Lindemann-Weierstrass) [Wei85] Let $z_1, \dots, z_n$ be n algebraic numbers linearly independent over $\mathbf{Q}$. Then $e^{z_1}, \dots, e^{z_n}$ are algebraically independent. Equivalently:*

$$\mathrm{tr.deg}_{\mathbf{Q}} \mathbf{Q}(e^{z_1}, \dots, e^{z_n}) = n$$

Note that in this statement, we can consider $(e^{z_1}, \dots, e^{z_n})$ as living in the algebraic group $G = ((\mathbf{C}^\times)^n, \times)$, and so being the exponential of the vector $(z_1, \dots, z_n)$, which lives in $(\mathbf{C}^n, +)$, the Lie algebra of G.

Now, instead of constant complex numbers, take $z_1, \dots, z_n$ to be rational functions $x_1(t), \dots, x_n(t)$. Then, generalize the Lindemann-Weierstrass statement to the case where G can be any algebraic subgroup, even defined over $\overline{\mathbf{C}(t)}$: this is the content of the Bertand-Pillay's theorem.

These notes are devoted to the classical framework of the problem; see Ana Peon Nieto's notes about the functional case. Our hope is to give an idea of the motivations around these questions, and describe how they are related to deeply geometric objects and statements such as abelian varieties, the Hodge conjecture or motives.

The first section will describe the classical context of periods of abelian varieties, defined as integrals of differential forms of first and second kind. We will there review the various notions of cohomology needed.

The second section will explain the construction of the Mumford-Tate group of an abelian variety (or, more properly, of a Hodge structure), and show that its dimension bounds the transcendence degree of the period extension. Grothendieck's and André's conjectures state reverse inequalities.

The third and last section will briefly expose the notion of 1-motive. Few precise definitions and no proof will be given there, but our goal is to explain how this extends the previous parts, and why we need these definitions to understand the Schanuel conjecture: we will end by showing that the Schanuel conjecture is equivalent to André's conjecture for a particular kind of 1-motives.

1 Abelian integrals and field of periods

1.1 Differential forms and periods

Let X be a projective smooth algebraic curve of genus g, defined over a subfield k of $\mathbf{C}$. The set of complex points $X(\mathbf{C})$ can be endowed with a differential structure: the associated differential variety will be denoted X^∞ (this is still true if X is of any dimension).

Choose a rational differential form ξ on X, and P and Q two points of X which are not poles of ξ. We are interested on integrals of the form:

$$\int_P^Q \xi$$

and transcendence degrees of field extensions of k generated by such values.

Obviously, this is not well defined: we have to specify a path* from P to Q, and we will restrict to some specific differential forms.

Usually, we will be able to talk about the integral $\int_\gamma \xi$ of a form ξ along a closed path γ (a loop, *i.e.* $P = Q$). To realize the previous integral as being of this kind, we will construct a new curve Y from X, by identifying P and Q: this process will transform a path from P to Q to a new closed one. However, we will loose both projectivity and smoothness: this leads to the notions of 1-*motive* and *motivic Galois group.*

Exemple 1.1. *To understand (roughly) the Schanuel conjecture in this context: take $x_1, \dots, x_n$ and their exponentials $y_1 = e^{x_1}, \dots, y_n = e^{x_n}$. Choose a determination of logarithm with $x_1 = \log y_1, \dots, x_n = \log y_n$. The attached variety is $\mathbf{G}_m \setminus \{y_1, \dots, y_n\}$ and x_i becomes a period by the well-known formula*

$$x_i = \int_1^{y_i} \frac{dt}{t}$$

Definition 1.2. *A rational differential form ξ is said to be:*

- exact *if ξ is the differential df of a rational function $f \in k(X)$. An exact form is of second kind (see below);*
- *of* first kind *if ξ is regular. Equivalently, ξ has no pole. The space of differential forms of first kind is $H^0(X, \Omega^1_X)$; it is a k-vector space of dimension g;*
- *of* second kind *if ξ is locally exact. Equivalently, if P is a pole of ξ and t a local parameter at P, $\xi = g(t)dt$, where g is a meromorphic function around t with residue zero at P. The space of differential forms of*

*Note that X^∞ is of dimension 2 as a differential variety over $\mathbf{R}$: a *path* is to be understood in the differential (or even topological) sense (a continuous path on a differential variety is always homotopic to a differential path).

second kind up to exact forms is $H^1_{dR}(X,k)$ *(the first algebraic de Rham cohomology group; see below); it is a* k*-vector space of dimension* $2g$*, of which* $H^0(X,\Omega^1_X)$ *is a subspace;*

- *of* third kind *if* ξ *is locally the sum of a logarithmic derivative (of the form* $n\frac{dt}{t}$ *with* $n \in \mathbf{Z}$*) and a regular form. Equivalently, the residual divisor of* ξ *is of the form* $\operatorname{Resdiv}\xi = \sum_i c_i(P_i)$*, where the* c_i*'s are integers, and the* P_i*'s are finitely many points on* X*.*

We will consider various spaces of cohomology. Fix a projective smooth algebraic variety M:

- the Betti (singular) cohomology: it is a topological invariant, related to the transcendental topology of M^∞. The Betti homology groups $H_{*,B}(M,\mathbf{Z})$ are the homology groups of the complex of simplices of M. The Betti cohomology groups $H^*_B(M,\mathbf{Z})$ are the $\mathbf{Z}$-duals of the homology. We will also consider the Betti (co)homology with rational coefficients, namely $H_{*,B}(M,\mathbf{Q}) = H_{*,B}(M,\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Q}$ and dually $H^*_B(M,\mathbf{Q}) = H^*_B(M,\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Q}$, or with complex coefficients. All these groups are finitely generated modules over the ring of coefficients ($\mathbf{Z}, \mathbf{Q}$ or $\mathbf{C}$);

- the differential de Rham cohomology: the $\mathbf{C}$-vector spaces $H^*(M^\infty,\mathbf{C})$ are the cohomology groups of the complex of C^∞-differential forms on M^∞ with values in $\mathbf{C}$, with exterior differential. For any n, $H^n(M^\infty,\mathbf{C})$ is the space of closed n-forms up to exact n-forms;

- the algebraic de Rham cohomology: the k-vector spaces $H^*_{dR}(M,k)$ are computed as the hypercohomology of the complex of Kähler differential forms on M.

A theorem of Grothendieck [Gro66] shows that we have a canonical isomorphism for each n:

$$H^n_{dR}(M,k) \otimes \mathbf{C} \simeq H^n(M^\infty,\mathbf{C})$$

If γ is an n-simplex of M, and M an n-differential form, we can define the integral $\int_\gamma \omega$. The Stokes' theorem says that it depends only on the classes of γ in $H_n(M,\mathbf{Z})$ and of ω in $H^n(M^\infty,\mathbf{C})$, so we get a pairing:

$$H_{n,B}(M,\mathbf{Z}) \times H^n(M^\infty,\mathbf{C}) \to \mathbf{C}$$

which extends to $\mathbf{C}$:

$$(H_{n,B}(M,\mathbf{Z}) \otimes \mathbf{C}) \times H^n(M^\infty,\mathbf{C}) \to \mathbf{C}$$

The de Rham theorem says that this last paring is perfect, so gives a canonical isomorphism

$$P^{(n)} : H^n_B(M,\mathbf{C}) \xrightarrow{\sim} H^n(M^\infty,\mathbf{C}) \simeq H^n_{dR}(M,k) \otimes_k \mathbf{C}$$

If Z is an algebraic cycle on M of codimension p, we have natural realisations $\mathrm{cl}_B(Z)$ and $\mathrm{cl}_{dR}(Z)$ of Z in the Betti and de Rham cohomologies of degree $2p$, and there is almost commutativity:

$$P^{(2p)}(\mathrm{cl}_B(Z)) = (2\pi i)^{2p}\mathrm{cl}_{dR}(Z)$$

Choose a $\mathbf{Q}$-basis of $H^n_B(M, \mathbf{Q})$, and a k-basis of $H^n_{dR}(M, k)$: the field extension L/k defined by the coefficients of the matrices of $P^{(n)}$ for the various n's is called the *field of periods* of M. We are thus interested on the *transcendence degree* of L/k. The philosophy will be to find an algebraic group whose dimension controls this transcendence degree.

If $M = A$ is an abelian variety, then for any cohomology, $H^n = \Lambda^n H^1$ and $P^{(n)} = \Lambda^n P^{(1)}$, so it is enough to look at the usual periods (defined in degree 1). Fix $(\gamma_1, \ldots, \gamma_{2g})$ a $\mathbf{Q}$-basis of $H^1_B(A, \mathbf{Q})$, $(\omega_1, \ldots, \omega_g)$ a k-basis of $H^0(A, \Omega^1_A)$ and $\eta_1, \ldots, \eta_g \in H^1_{dR}(A, k)$ such that $(\omega_1, \ldots, \omega_g, \eta_1, \ldots, \eta_g)$ gives a k-basis of $H^1_{dR}(A, k)$: we look at the transcendence degree over k of the integrals:

$$\int_{\gamma_i} \omega_j, \int_{\gamma_i} \eta_j \in \mathbf{C}$$

The $(2g \times 2g)$-matrix will be called the *matrix of periods.*

Exemple 1.3. *Let E be the elliptic curve ($g = 1$) defined by the homogeneous equation in the projective plane:*

$$y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3$$

Let $\wp(z)$ be the Weierstrass function of E, and Λ the lattice of $\mathbf{C}$ such that $z \mapsto (\wp(z), \wp'(z), 1)$ gives an isomorphism $\mathbf{C}/\Lambda \xrightarrow{\sim} \to E(\mathbf{C})$.

Take (γ_1, γ_2) a positive basis of Λ. The differential $\omega = \frac{dx}{y}$ on E pulls back to dz on $\mathbf{C}$, and then is of first kind. The differential $\eta = \frac{xdx}{y}$ corresponds to $\wp(z)dz$ on $\mathbf{C}$, hence is of second kind.

If we note $\omega_i = \int_{\gamma_i} \omega$ and $\eta_i = \int_{\gamma_i} \eta$ ($i = 1, 2$), then we can check that the matrix of periods has determinant $2\pi i$, i.e. *we have the Legendre relation:*

$$\omega_1 \eta_2 - \omega_2 \eta_1 = 2\pi i$$

1.2 Differential forms of first kind on a curve: the Abel-Jacobi map

Fix a smooth and projective curve X defined over $k \subset \mathbf{C}$ of genus g, and P_0 a k-point of X.

We can consider the Jacobian variety J of X, in various ways: the group $\mathrm{Pic}^0(X)$ of invertible sheaves (line bundles) of degree zero on X up to linear equivalence (or, equivalently, because X is smooth, the group of divisors

$\sum_i n_i P_i$ of degree zero), can be endowed with a canonical structure of algebraic group over k. The canonical morphism associated to P_0 is defined on $X(k)$ by

$$f^{P_0} : P \mapsto (P) - (P_0)$$

Over the field of complex numbers $\mathbf{C}$, take a basis $(\omega_1, \ldots, \omega_g)$ of $H^0(X, \Omega^1_X)$ (regular forms), and $\gamma_1, \ldots, \gamma_{2g}$ a basis of the free $\mathbf{Z}$-module $H_{1,B}(X, \mathbf{Z})$. The vectors $\left(\int_{\gamma_j} \omega_i\right)_{1 \leqslant i \leqslant g}$ generate a lattice Ω in $\mathbf{C}^g$: J is defined to be the complex torus $\mathbf{C}^g/\Omega$. The canonical morphism associated to P_0 is then, on $X(\mathbf{C})$ †:

$$f^{P_0} : P \mapsto \left(\int_{P_0}^{P} \omega_1, \ldots, \int_{P_0}^{P} \omega_g\right) \bmod \Omega$$

Recall that on an abelian variety A, any regular differential form (an element of $H^0(A, \Omega^1_A)$) is invariant: it is fixed under the natural group action of A.

The Abel-Jacobi theorem says that:

$$H^0(X, \Omega^1_X) = H^0(J, \Omega^1_J)$$

So to understand transcendence degrees of periods defined by regular forms on X, we can look at its jacobian J.

1.3 Differential forms of second kind on a curve; Severi-Weil-Rosenlicht

As we saw previously, going from the curve X to its jacobian J allows us to understand the differential forms of first kind on X as invariant forms on an abelian variety of dimension g.

The extensions of periods of X and of J are the same. A form of second kind on X is the pull-back of a form of second kind on J, but the latter is not invariant (otherwise, it would be regular: by transitivity of the action of J, if there is one pole, every point is also a pole). The following construction gives an algebraic group on which we see a differential form of second kind η as an invariant form.

Fix an abelian variety A of dimension g: following the previous discussion, it can be the Jacobian of a curve of genus g.

Consider the algebraic vector group $\mathbf{G}_a$ as equipped with a "canonical" invariant form dt (which is unique up to a non-zero scalar).

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†We have to check that it is well defined: for $P \in X(\mathbf{C})$, choose two paths γ, γ' from P_0 to P. Then $\alpha = \gamma \cdot (\gamma')^{-1}$ is a loop around P_0: by definition of Ω,

$$\left(\int_{\gamma} \omega_1, \ldots, \int_{\gamma} \omega_g\right) - \left(\int_{\gamma'} \omega_1, \ldots, \int_{\gamma'} \omega_g\right) = \left(\int_{\alpha} \omega_1, \ldots, \int_{\alpha} \omega_g\right) \in \Omega$$

Theorem 1.4. *[Ser84] Let η be a differential form of second kind on A. There exists a unique extension of A by $\mathbf{G}_a$ noted $E_\eta \in \mathrm{Ext}(A, \mathbf{G}_a)$ (in the category of algebraic groups) which has the following property:*

if σ is any rational section of $E_\eta \to A$ (whose existence is given by Hilbert 90), then there exists a unique $\widetilde{\eta} \in H^0(E_\eta, \Omega^{1,inv}_{E_\eta})$ (regular invariant forms) such that there exists $f \in k(A)$ which satisfies (with $i : \mathbf{G}_a \to E_\eta$ the injection) :

$$i^*\widetilde{\eta} = dt$$

$$\sigma^*\widetilde{\eta} = \eta + df$$

Remark 1.5. *Choose another section σ': then there exists a rational map $\varphi : A \to \mathbf{G}_a$ such that $\sigma' = \sigma + i \circ \varphi$. Then: ${\sigma'}^*\widetilde{\eta} = \sigma^*\widetilde{\eta} + \varphi^*(i^*\widetilde{\eta})$ and $\varphi^*(i^*\widetilde{\eta}) = \varphi^* dt = d\varphi$, so*

$${\sigma'}^*\widetilde{\eta} = \sigma^*\widetilde{\eta} + d\varphi$$

So $\widetilde{\eta}$ does not really depend on the choice of σ.

Remark 1.6. *The class of E_η in $\mathrm{Ext}(A, \mathbf{G}_a)$ depends only on the class of $\eta \in H^1_{dR}(A, k)$ modulo exact and invariant forms.*

For example, if $\eta = df$, with $f \in k(A)$: $E_\eta = \mathbf{G}_a \times A$ is the trivial extension, $\sigma(a) = (f(a), a)$ and $\widetilde{\eta} = \phi^* dt$ (where $\pi : E_\eta \to \mathbf{G}_a$ is the projection).

Remark 1.7. *If η is a form of second kind on A, P, Q two points: using the previous notations, we get (with the change of variables formula)*

$$\int_{\sigma(P)}^{\sigma(Q)} \widetilde{\eta} = \int_P^Q \eta + f(P) - f(Q)$$

Since $f(P) - f(Q) \in k$, going from A to E_η does not change anything on the transcendence degrees of integrals.

So far, we have constructed a map from $H^1_{dR}(A, k)$ to $\mathrm{Ext}(A, \mathbf{G}_a)$ that associates to η the extension E_η, and which is trivial on the invariant form on A.

Proposition 1.8. *This is a morphism of groups, whose kernel is exactly $H^0(A, \Omega^1_A)$.*

Theorem 1.9. *The canonical isomorphism* [‡]

$$H^1_{dR}(A,k)/H^0(A,\Omega^1_A) \simeq H^1(A,\mathcal{O}_A)$$

induces an isomorphism:

$$H^1(A,\mathcal{O}_A) \xrightarrow{\sim} \mathrm{Ext}(A,\mathbf{G}_a)$$

We can then construct a *universal vectorial extension* $\widetilde{A}$. Recall that a vectorial group is an algebraic group isomorphic to a product $\mathbf{G}_a^n$.

Definition 1.10. *A* universal vectorial extension $\widetilde{A}$ *of A is an extension of A by a vectorial group U such that: for any vectorial group V, any extension $E \in$* Ext(A,V), *there is a unique $f : U \to V$ such that $E = f_*(\widetilde{A})$,* i.e. *there is an $\tilde{f}$ (that happens to be unique) with the following commutative diagram:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U & \longrightarrow & \widetilde{A} & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \tilde{f}} & & \| & & \\
0 & \longrightarrow & V & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

Set

$$W_A = H^1(A,\mathcal{O}_A)^\vee$$

(the dual algebraic group).

Proposition 1.11. *The abelian variety A has a universal vectorial extension $\widetilde{A}$, which is an extension of A by W_A.*

For any differential form of second kind η (which can be seen as a linear form $\eta : W_A \to \mathbf{G}_a$), we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W_A & \xrightarrow{i} & \widetilde{A} & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \eta} & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathbf{G}_a & \longrightarrow & E_\eta & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

The projection $\widetilde{A} \to A$ admits a rational section $\widetilde{A} \overset{\sigma}{\leftrightarrows} A$ such that:

for any η, there exists a unique $\widetilde{\eta} \in H^0(\widetilde{A}, \Omega^{1,inv}_{\widetilde{A}})$ with $\sigma^*\widetilde{\eta} = \eta$ up to an exact form and $i^*\widetilde{\eta} = \eta^* dt$.

[‡]Take A to be the Jacobian of the curve X: the exact sequence of sheaves

$$O \to \mathbf{C} \to \mathcal{O}_X \to \Omega^1_X \to 0$$

defined by differentiation gives rise to a long exact sequence of cohomology:

$$H^0(X,\mathcal{O}_X) \to H^0(X,\Omega^1_X) \to H^1(X,\mathbf{C}) \to H^1(X,\mathcal{O}_X) \to H^1(X,\Omega^1_X) \to H^2(X,\mathbf{C})$$

The first arrow is differentiation of constant functions (because X is proper) hence is zero. The last arrow is an isomorphism $H^1(X,\Omega^1_X) \simeq H^0(X,\mathcal{O}_X)^\vee \simeq \mathbf{C} \to H^2(X,\mathbf{C}) \simeq \mathbf{C}$ (the first isomorphism is given by Serre's duality) so the map $H^1(X,\mathcal{O}_X) \to H^1(X,\Omega^1_X)$ is zero, and the map $H^1(X,\mathbf{C}) \to H^1(X,\mathcal{O}_X)$ is surjective.

$\widetilde{A}$ is an algebraic group (no more an abelian variety than the E_η's are), extension of A by W_A, of dimension $2g$.

It allows now us to understand any differential form of second kind on our curve X as an invariant regular form on an algebraic group. The field of periods of X (or of A) is exactly the field generated by the integrals on $\widetilde{A}$ of invariant (regular) forms, and the matrix of periods is given by choosing a basis of invariant forms on $\widetilde{A}$.

The construction described in this section can be extended to the study of all differentials of X: it suffices to replace A and its universal extension $\widetilde{A}$ by a semiabelian variety A' and its universal extension $\widetilde{A'}$, where A' is an extension of A of by a suitable torus $\mathbf{G}_m^s$.

2 The Mumford-Tate group

We get back to the case of an *abelian* variety A with field of periods L. As we already mentioned, the general case is described by the theory of motives and motivic Galois groups, see [And04].

The idea, due to Grothendieck, is that the existence of cycles of a specific kind (algebraic cyles, Hodge cycles, motivated cycles) can force algebraic relations between the periods of abelian integrals. It leads to define in each case an algebraic group whose dimension will, conjecturally, control the transcendence degree of the field of periods.

The first group we define below (in the context of abelian varieties) only takes into account the algebraic cycles: we will call it the "naive" group. We then define the Mumford-Tate group, defined in the same way but using Hodge cycles instead of algebraic ones. We will need here a theorem of Deligne that explains how Hodge cycles give the relations between periods for an abelian variety. André's construction of the motivic Galois group relies on the notion of motivated cycles.

In the following, unless otherwise specified, $H^n_B(A)$ will always denote $H^n_B(A, \mathbf{Q})$ and $H^n_{dR}(A)$ will always denote $H^n_{dR}(A, k)$.

2.1 A naive group

Recall that for any integer n, $H^n(A) = \bigwedge^n H^1(A)$, $P^{(n)} = \bigwedge^n P^{(1)}$ and $H^1(A^s) = \bigoplus_{i=1}^s H^1(A)$ for any cohomology theory.

As we know, the compatibility between comparison theorems and cycle maps has to take account of the following relation:

$$P^{(2p)}(\mathrm{cl}_B(Z)) = (2\pi i)^{2p} \mathrm{cl}_{dR}(Z)$$

To avoid endless questions about this factor $2\pi i$, is it convenient to introduce the notion of *Tate twist*.

Define $\mathbf{Q}_B(1) = 2\pi i\mathbf{Q}, \mathbf{Q}_{dR}(1) = k$, and then, for each $m \in \mathbf{Z}, n \geqslant 0$:

$\mathbf{Q}_B(m) = \mathbf{Q}_B(1)^{\otimes m}$

$H^n_B(A)(m) = H^n_B(A) \otimes_{\mathbf{Q}} \mathbf{Q}_B(m)$

$\mathbf{Q}_{dR}(m) = \mathbf{Q}_{dR}(1)^{\otimes m}$

$H^n_{dR}(A)(m) = H^n_{dR}(A) \otimes_k \mathbf{Q}_{dR}(m)$ (note that the Tate twist in this case is trivial).

The natural definition of cl_B sends actually a cycle Z of codimension p in $H^{2p}_B(A)(p)$, the comparison isomorphism naturally extends to:

$H^n_B(A)(m) \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} H^n_{dR}(A)(m) \otimes_k \mathbf{C}$ and the class maps are now compatible with these isomorphisms.

Definition 2.1.
- *Let $\lambda \in \mathbf{G}_m(\mathbf{Q})$ acts on $\mathbf{Q}_B(1)$ as λ^{-1}. There is a natural action of $GL(H^1_B(A)) \times \mathbf{G}_m$ on $H^n_B(A^s)(m)$ for any n, s, m. Define G to be the subgroup of $GL(H^1_B(A)) \times \mathbf{G}_m$ fixing all tensors of the form $\mathrm{cl}_B(Z)$ for Z a cycle on some A^s.*

  G is an algebraic group defined over $\mathbf{Q}$.

- *For any k-algebra R, define $\mathcal{T}(R)$ to be the set of isomorphisms of R-modules $\varphi : H^1_B(A) \otimes_{\mathbf{Q}} R \xrightarrow{\sim} H^1_{dR}(A) \otimes_k R$ mapping $\mathrm{cl}_B(Z) \otimes 1$ to $\mathrm{cl}_{dR}(Z) \otimes 1$ for any cycle Z on a power of A.*

  $\mathcal{T}$ is the functor of points of a scheme over k.

Lemma 2.2. *$G \otimes_{\mathbf{Q}} k$ acts on $\mathcal{T}$ on the left by composition. By this action, $\mathcal{T}$ becomes a torsor under $G \otimes_{\mathbf{Q}} k$.*

The crucial fact is that $P^{(1)}$ is a $\mathbf{C}$-point of $\mathcal{T}$, with field of definition L.

Theorem 2.3. *We have the following inequality, for k any subfield of $\mathbf{C}$:*

$$\mathrm{tr.deg}_k L \leqslant \dim g$$

Proof. $\mathcal{T}$ is a torsor under $G \otimes_{\mathbf{Q}} k$, so $\dim \mathcal{T} = \dim G$. $P^{(1)}$ is a point of $\mathcal{T}$ with field of definition L, so $\mathrm{tr.deg}_k L \leqslant \dim \mathcal{T}$. □

Exemple 2.4. *If E is an elliptic curve, the $\dim G = 2$ if E has complex multiplication (*$\mathrm{End}(E)$ *is stricly larger than* $\mathbf{Z}$*), and $\dim G = 4$ otherwise. It is known (see [Chu84]) that equality holds in the first case (and it is actually the only situation where the following conjecture is known to be true).*

Conjecture 2. *(Grothendieck) If $k \subset \overline{Q}$, $P^{(1)}$ is a generic point of $\mathcal{T}$,* i.e. *we have an equality:*

$$\mathrm{tr.deg}_k L = \dim G$$

Conjecture 3. *(André) If k is any subfield of* $\mathbf{C}$, *we have an inequality:*

$$\mathrm{tr.deg}_{\mathbf{Q}} L \geqslant \dim G$$

2.2 Construction of the Mumford-Tate group

The Mumford-Tate group is formally the same as the "naive" group described in the preceding section, but using *Hodge cycles* instead of algebraic cycles.

An algebraic cycle always being a Hodge cycle, the Mumford-Tate group will be potentially smaller, and the conjecture of Grothendieck (with algebraic coefficients) will then be sharper. Obviously, if the Hodge conjecture is true, we will get the same cycles, and the same groups.

Definition 2.5. *Let V be a finite-dimensional vector space over the field $\mathbf{Q}$. A $\mathbf{Q}$-rational* Hodge structure *of weight n on V is a decomposition of the $\mathbf{C}$-vector space $V_{\mathbf{C}} = \bigoplus_{p+q=n} V^{p,q}$ where $\overline{V^{p,q}} = V^{q,p}$.*

The theory of Kähler manifolds shows that the Betti cohomology of degree n of A has a canonical $\mathbf{Q}$-rational Hodge structure of weight n.

Denote $V = H^1_B(A, \mathbf{Q})$. For any $m_1, m_2 \geqslant 0$, $m_3 \in \mathbf{Z}$, the tensor product $T = V^{\otimes m_1} \otimes (V^\vee)^{\otimes m_2} \otimes \mathbf{Q}(1)^{\otimes m_3}$ has a Hodge structure of weight $m_1 - m_2 - 2m_3$. An element of $T_{\mathbf{C}}$ is *rational of bidegree (p,q)* if it lies in $T \cap T^{p,q}$.

Let $\lambda \in \mathbf{G}_m$ act on $\mathbf{Q}(1)$ as λ^{-1}: we have a natural action of $GL(V) \times \mathbf{G}_m$ on any T.

Definition 2.6. *The Mumford-Tate group G_{MT} of A is the subgroup of $GL(V) \times \mathbf{G}_m$ fixing all rational tensors of bidegree $(0,0)$ belonging to any T.*

G_{MT} is an algebraic group over $\mathbf{Q}$.

Equivalently, $G(\mathbf{Q})$ is the set of all $g \in GL(V)$ such that: there exists $\lambda \in \mathbf{Q}^*$ (depending on g) with the property that $g(t) = \lambda^p t$ for any Hodge cycle $t \in V^{\otimes m_1} \otimes (V^\vee)^{\otimes m_2}$ of bidegree (p,p).

Because any algebraic cycle is Hodge, the Mumford-Tate group is a subgroup of the "naive" group G previously defined.

Theorem 2.7. *(Deligne) The following inequality holds, for k any subfield of $\mathbf{C}$:*

$$\text{tr.deg}_k L \leqslant \dim G_{MT}$$

To show this theorem, we want to mimic the proof of the previous paragraph. For this, we need to define a k-scheme $\mathcal{T}_{MT}$ that will be a torsor under $G_{MT} \otimes k$, and with the property that the the period isomorphism $P^{(1)}$ is a $\mathbf{C}$-point of $\mathcal{T}_{MT}$.

For any k-algebra R, we would like to define $\mathcal{T}_{MT}(R)$ to be the set of isomorphisms $\varphi : H^1_B(A) \otimes_{\mathbf{Q}} R \xrightarrow{\sim} H^1_{dR}(A) \otimes_k R$ such that φ sends a Hodge cycle to its de Rham class. Unfortunately, we do not have such a class! We only know that a Hodge cycle (an object living in some $H^{2p}_B(A)(p)$ has *a priori* an image in $H^{2p}_{dR}(A)(p) \otimes_k \mathbf{C}$, given by the period isomorphism. However, in the case of an abelian variety, Deligned shows in [DMOS82] the following:

Definition 2.8. *For $\sigma : k \to \mathbf{C}$ an embedding, say that $t \in H^{2p}_{dR}(A)(p)$ is a* Hodge cycle relative to σ *if it is the image of a Hodge cycle by the period isomorphism*[§¶].

An element $t \in H^{2p}_{dR}(A)(p)$ is an absolute Hodge cycle *if it is a Hodge cycle relatively to any σ.*

Theorem 2.9. *(Deligne)[DMOS82] If A is an abelian variety, a Hodge cycle $t \in H^{2p}_{dR}(A)(p)$ relative to one embedding of k in $\mathbf{C}$ is an absolute Hodge cycle.*

*Moreover, if t is a Hodge cycle in $H^{2p}_{B}(A)(p)$, its image by the period isomorphism always lies in $H^{2p}_{dR}(A)(p)$ (*i.e. *is defined over k).*

So, using the theorem, we can talk of *the* de Rham class of a Hodge cycle: if $t \in H^{2p}_{B}(A)(p)$ is a Hodge cycle, it is its image in the de Rham cohomology, given by the period isomorphism, but in fact living over k.

For a k-algebra R, let $\mathcal{T}_{MT}(R)$ be the set of isomorphisms of R-modules $\varphi : H^1_B(A) \otimes_{\mathbf{Q}} R \xrightarrow{\sim} H^1_{dR}(A) \otimes_k R$ such that φ sends a Hodge cycle to its de Rham class. As previously, $\mathcal{T}_{MT}$ is a k-scheme wich is a torsor under $G_{MT} \otimes k$, and $P^{(1)}$ is a $\mathbf{C}$-point of $\mathcal{T}$ with residual field L. This concludes the proof of the inequality.

As before, we can formulate two conjectures:

Conjecture 4. *(Grothendieck) If $k \subset \overline{Q}$, $P^{(1)}$ is a generic point of $\mathcal{T}$,* i.e. *we have an equality:*

$$\mathrm{tr.deg}_k L = \dim G_{MT}$$

Conjecture 5. *(André) If k is any subfield of $\mathbf{C}$, we have an inequality:*

$$\mathrm{tr.deg}_{\mathbf{Q}} L \geqslant \dim G_{MT}$$

[§]So far, we have dealt with the Betti cohomology as if it was unique. But it depends only on the transcendental topology on the variety: in particular, for *any* embedding $k \to \mathbf{C}$ we have a possibly new Betti cohomology with the associated period isomorphism, even if the field k is given as a *subfield* of $\mathbf{C}$, and a distinguished embedding is part of the data. Eventually, the theorem of Deligne shows that we do not have to be so careful.

[¶]Note that we ask the cycle t to be defined *over k*: this is *a priori* a very strong requirement if we do not want to assume the Hodge conjecture.

3 1-motives

As we explained at the beginning of this text, to understand the Schanuel conjecture in terms of periods, we have to consider more general objects than smooth projective varieties. The notion of 1-*motive* will give the answer. Most of the definitions will not be given here, but we will try to give an idea of why we consider such objects. The reader who wants precise definitions can look at [Del74, DMOS82, Ber02, And04].

Definition 3.1. *A* 1-motive $M = [\Lambda \xrightarrow{f} B]$ *over the field k is the following data:*

- *a free* $\mathbf{Z}$*-module finitely generated* Λ*,*
- *a semi-abelian variety B (extension of an abelian variety A by a torus T),*
- *a group homomorphism* $u : \Lambda \to B(K)$*.*

Exemple 3.2. *If X is a smooth projective curve, the* 1*-motive of X is given by* $\Lambda = 0$ *and the Jacobian of X.*

We note $M_{\mathbf{C}}$ the 1-motive defined over $\mathbf{C}$ by $(\Lambda, A_{\mathbf{C}}, T_{\mathbf{C}}, B_{\mathbf{C}}, u)$. $T_{\mathbf{Z}}(M_{\mathbf{C}})$ is the fibered product of the Lie algebra $LB_{\mathbf{C}}$ of $B_{\mathbf{C}}$ and Λ above $B_{\mathbf{C}}$:

$$T_{\mathbf{Z}}(M_{\mathbf{C}}) = \{(b, \lambda) \in LB_{\mathbf{C}} \times \Lambda : \exp(b) = u(\lambda)\}$$

The *Hodge realization* of M is the $\mathbf{Q}$-vector space $T_H(M_{\mathbf{C}}) = T_{\mathbf{Z}}(M_{\mathbf{C}}) \otimes_{\mathbf{Z}} \mathbf{C}$.

We have here a canonical (mixed) $\mathbf{Q}$-Hodge structure, which allows us to define the *Mumford-Tate group G_{MT} of M*, which is an algebraic group over $\mathbf{Q}$.

We can also define a k-vector space $T_{dR}(M)$ called *the de Rham realization* of M. With $H_H(M_{\mathbf{C}}) = \mathrm{Hom}_{\mathbf{Q}}(T_H(M_{\mathbf{C}}), \mathbf{Q})$ (check this is the Betti cohomology for an abelian variety!) and $H_{dR}(M) = \mathrm{Hom}_k(T_{dR}(M), k)$, one shows the following:

Proposition 3.3. *There is a canonical isomorphism of* $\mathbf{C}$*-vector spaces:*

$$P : H_{dR}(M) \otimes_k \mathbf{C} \xrightarrow{\sim} H_H(M_{\mathbf{C}}) \otimes_{\mathbf{Q}} \mathbf{C}$$

P is called the *period isomorphism*, and a matrix of P in bases of $H_{dR}(M)$ over k and of $H_H(M_{\mathbf{C}})$ over $\mathbf{Q}$ is called a *period matrix*. The field extension L of k given by the coefficients does not depend on the choice of bases, and is called the *field of periods* of M.

Brylinski shows in [Bry83] that Hodge cycles are absolute in this context, which gives the following result for 1-motives:

Theorem 3.4. *if M is a 1-motive defined over k, then we have the following inequality:*

$$\mathrm{tr.deg}_k L \leqslant \dim G_{MT}$$

We can then, again, formulate our two conjectures:

Conjecture 6. *(Grothendieck) For a 1-motive M defined over k a subfield of $\overline{\mathbf{Q}}$:*

$$\mathrm{tr.deg}_k L = \dim G_{MT}$$

Conjecture 7. *(André) For a 1-motive M defined over k a subfield of $\mathbf{C}$:*

$$\mathrm{tr.deg}_{\mathbf{Q}} L \geqslant \dim G_{MT}$$

The connexion with the Schanuel conjecture is given by the following two lemmas:

Lemma 3.5. *Let k be a subfield of $\mathbf{C}$, $r, s \geqslant 0$, $q_{ij} \in \mathbf{G}_m(k)$ for any $1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s$, and $M = [\mathbf{Z}^r \xrightarrow{u} \mathbf{G}_m^s]$ the 1-motive defined by $u(z_i) = (q_{i1}, \dots, q_{is}) \in \mathbf{G}_m^s(k)$, where $(z_1, \dots, z_r)$ is the canonical basis of $\mathbf{Z}^r$. A matrix of periods of M is of the following form:*

$$\begin{pmatrix} & \log q_{11} & \dots & \log q_{1s} \\ \mathrm{Id}_{r\times r} & \vdots & \ddots & \vdots \\ & \log q_{r1} & \dots & \log q_{rs} \\ & & & \\ 0 & & 2\pi i \mathrm{Id}_{s\times s} & \\ & & & \end{pmatrix}$$

Here, $\log q_{ij}$ is an arbitarily chosen determination of the logarithm of the point q_{ij}.

Lemma 3.6. *With the previous notations: define Z to be the subgroup of $\mathbf{G}_m(k)$ generated by the q_{ij}. Then:*

$$\dim G_{MT} = 1 + \mathrm{rk}\, Z$$

Proposition 3.7. *André's conjecture for 1-motives of the previous kind and the Schanuel conjecture are equivalent.*

Proof. Assume André's conjecture: take $x_1, \dots, x_s \in \mathbf{C}$ that are $\mathbf{Q}$-linearly independent.

If $2\pi i$ is in the $\mathbf{Q}$-subspace they generate: we can assume $x_s = 2\pi i$. Then the 1-motive $M = [\mathbf{Z} \xrightarrow{u} \mathbf{G}_m^{s-1}]$ defined by $u(1) = (e^{x_1}, \dots, e^{x_{s-1}})$ answers the question: its periods are $x_1, \dots, x_{s-1}, 2\pi i$, and by the previous lemma $\dim G_{MT} = s$. Conclude with André's conjecture for M.

Otherwise, $x_1, \dots, x_s, 2\pi i$ are $\mathbf{Q}$-linearly independent. Take the 1-motive $M = [\mathbf{Z} \xrightarrow{u} \mathbf{G}_m^s]$ defined by $u(1) = (e^{x_1}, \dots, e^{x_s})$. The periods of M are

$x_1, \ldots, x_s, 2\pi i$ and by the previous lemma, $\dim G_{MT} = s+1$. Again, André's conjecture clearly concludes.

Conversely, assume the Schanuel conjecture. The periods of the 1-motive $M = [\mathbf{Z} \xrightarrow{u} \mathbf{G}_m^s]$ defined by $u(1) = (q_1, \ldots, q_s)$ are $\log q_1, \ldots, \log q_s, 2\pi i$, where the $\log q_i$'s are arbitrary complex numbers such that $e^{\log q_i} = q_i$. The field of periods of M is generated over $\mathbf{Q}$ by $\log q_1, \ldots, \log q_s, 2\pi i$. Let V be the $\mathbf{Q}$-subspace of $\mathbf{C}$ generated by these numbers. If $(x_1, \ldots, x_d)$ is a $\mathbf{Q}$-basis of V, then we can apply the Schanuel conjecture to the x_i's, which gives: $\text{tr.deg}_{\mathbf{Q}} L = \text{tr.deg}_{\mathbf{Q}} \mathbf{Q}(x_1, \ldots, x_d, e^{x_1}, \ldots, e^{x_d}) \geqslant d$ (with L the field of periods).

Now, the ranks of the subgroups of $\mathbf{G}_m$ generated by the q_i's on the one hand, and by the e^{x_i}'s on the other hand, coincide. The latter is equal to $d-1$ (because $2\pi i$ lies in V), and we finally derive André's conjecture from lemma 3.6. □

References

[And04] Yves André. *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, volume 17 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 2004.

[Ber02] Cristiana Bertolin. Périodes de 1-motifs et transcendance. *J. Number Theory*, 97(2):204–221, 2002.

[Bry83] Jean-Luc Brylinski. "1-motifs" et formes automorphes (théorie arithmétique des domaines de Siegel). In *Conference on automorphic theory (Dijon, 1981)*, volume 15 of *Publ. Math. Univ. Paris VII*, pages 43–106. Univ. Paris VII, Paris, 1983.

[Chu84] Gregory V. Chudnovsky. Measure of the algebraic independence of periods and quasiperiods of elliptic curves. In *Contributions to the theory of transcendental numbers*, volume 19 of *Math. Surveys Monogr.*, pages 355–382. Amer. Math. Soc., Providence, RI, 1984.

[Del74] Pierre Deligne. Théorie de Hodge. III. *Inst. Hautes Études Sci. Publ. Math.*, (44):5–77, 1974.

[DMOS82] Pierre Deligne, James S. Milne, Arthur Ogus, and Kuang-yen Shih. *Hodge cycles, motives, and Shimura varieties*, volume 900 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1982.

[Gro66] A. Grothendieck. On the de Rham cohomology of algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.*, (29):95–103, 1966.

[Ser84] Jean-Pierre Serre. *Groupes algébriques et corps de classes.* Publications de l'Institut Mathématique de l'Université de Nancago [Publications of the Mathematical Institute of the University of Nancago], 7. Hermann, Paris, second edition, 1984. Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics], 1264.

[Wei85] C. Weierstrass. Zu Lindemann's Abhandlung: "Über die Ludolph'sche Zahl". *Berl. Ber. 1067-1086*, (1885).