O-minimal structures

**Definition**

A structure $M = (M, <, \ldots )$ is **o-minimal** if every definable subset of $M$ is a finite union of points and intervals $(a, b)$ with $a, b \in M \cup \{\pm \infty\}$.

**Example**

1. $(\mathbb{R}, <, +, \cdot )$.
2. Any real closed field $(R, <, +, \cdot )$.
3. $(\mathbb{R}, <, +, \cdot , \exp, \sin_{[0,1]} )$
4. $\mathbb{R}_{an}$
Triangulations

Fix an o-minimal structure $M$. We always assume that $M$ expands an ordered field. We recall the following basic result.

**Theorem ([vdD98])**

Every definable set $X \subseteq M^n$ can be triangulated, namely there is a finite (open) simplicial complex $K$ and a definable homeomorphism $f : |K|^M \to X$.

In general two compact polyhedra $|K|$ and $|L|$ can be homeomorphic without being PL-homeomorphic. However:

**Theorem (O-minimal Hauptvermutung: [Shi10])**

Let $K, L$ be finite closed simplicial complexes and let $f : |K|^M \to |L|^M$ be a definable homeomorphism. Then $K$ and $L$ have isomorphic subdivisions. So there is a PL-homeomorphism $g : |K|^M \cong |L|^M$.

Definable groups

**Definition**

A definable group is a definable set $G \subseteq M^n$ with a definable group operation.

**Example**

$$SO_2(M) = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a^2 + b^2 = 1 \right\}.$$  

**Example**

A non singular elliptic curve $y^2 = x^3 + ax + b$ in $\mathbb{P}^2(M)$ ("real" case) or in $\mathbb{P}^2(M[\sqrt{-1}])$ ("complex" case).
Tori

Lie tori
Every connected compact abelian real Lie group is Lie-isomorphic to a torus \( \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n \).

In the o-minimal context \( \mathbb{Z} \) does not exist, but we can define:

Definable tori
Let \( \mathbb{T}^1(M) = [0, 1) \subset M \) with the following group operation:

\[
x * y = \begin{cases} 
  x + y & \text{if } x + y < 1 \\
  x + y - 1 & \text{otherwise}
\end{cases}
\]

Similarly we define the \( n \)-th torus \( \mathbb{T}^n = [0, 1)^n \).

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t-topology

Theorem ([Pil88])
Any definable group \((G, \ast)\) admits a unique group topology, called the t-topology, such that \( G \) is a finite union \( U_1 \cup \ldots \cup U_n \) with each \( U_i \) open in \( G \) and definably homeomorphic to \( M^n \) \((n = \dim(G))\).

So when \( M = (\mathbb{R}, <, \ldots) \), any definable group \( G \) is a real Lie group.

Example
With the t-topology the torus \( \mathbb{T}^1 = [0, 1) \) (with addition modulo 1) is definably homeomorphic to \( S^1 \) (the unit circle in \( M \)), so the t-topology does not coincide with the subspace topology \( [0, 1) \subset \mathbb{R} \).
t-topology

**Definition**

*G* is **definably compact** if every definable curve \( f : (0, \varepsilon) \to G \) has a limit in \( G \) in the t-topology.

*G* is **definably connected** if \( G \) has no proper definable clopen subset in the t-topology. This is equivalent to say that \( G \) has not definable subgroups of finite index.

By “Robson’s embedding theorem” we have:

**Fact**

*Every definable group \( G \) can be embedded in some \( M^m \), namely \( G \) is definably isomorphic to a group \( G' \subset M^m \) such that the t-topology of \( G' \) coincides with the subspace topology inherited from \( M^m \).*

Such a \( G' \) is definably compact iff it is closed and bounded in \( M^m \).

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**One-dimensional definable groups**

We have seen various examples of one-dimensional definable groups: \( SO(2, M) \), \( T^1(M) \), elliptic curves in \( \mathbb{P}^2(M) \).

Note that \( SO(2, \mathbb{R}) \) is Lie-isomorphic to \( T^1(\mathbb{R}) \) but not definably isomorphic in \( (\mathbb{R}, <, +, \cdot) \) (i.e. the isomorphism is not semialgebraic).

**Theorem ([Str94b])**

*Let \( G \) be a definably compact definably connected abelian definable group in \( M \). Assume \( \text{dim}(G) = 1 \). Then \( G \), with the t-topology, is definably homeomorphic to \( T^1 \) (equivalently: to the unit circle \( S^1 \)).*

A natural conjecture is that when \( \text{dim}(G) = n \), then \( G \) is definably homeomorphic to \( T^n \). The difficulty is that \( G \) may have no one-dimensional definable subgroups ([PS99]).
Main result

Theorem ([BB11])

Let $G$ be a definably compact definably connected abelian $n$-dimensional definable group with $n \neq 4$. Then $G$ is definably homeomorphic to $\mathbb{T}^n$. In the semialgebraic case the proviso $n \neq 4$ is not needed.

Main steps.

1. The result holds in the semialgebraic case;
2. Homotopy transfer;
3. $\pi_1(G) \cong \mathbb{Z}^n \cong \pi_1(\mathbb{T}^n)$ ([EO04]);
4. $\pi_k(G) = 0$ for $k > 0$;
   $G$ is definably homotopy equivalent to $\mathbb{T}^n$ ([BMO10]);
5. If $\text{dim}(G) \neq 4$, there is a finite cover $f : G \rightarrow \mathbb{T}^n$ (as spaces). Use $f$ to put a semialgebraic group operation on $\text{dom}(G)$.

Step 1: Semialgebraic case

Lemma (Elimination of parameters)

Let $G$ be a semialgebraic group over a real closed field $M$. Then $G$ is semialgebraic homeomorphic (but not necessarily isomorphic) to a semialgebraic group over $\overline{\mathbb{Q}}^{\text{real}} \prec M$.

Proof.

- We can assume that the $t$-topology of $G$ is the topology inherited from the ambient space $M^m$ (Robson’s embedding theorem).
- By the triangulation theorem we can assume that $\text{dom}(G) = |K|$ where $K$ is a finite simplicial complex definable without parameters.
- However the group operation may need parameters from $M$.
- Since $\overline{\mathbb{Q}}^{\text{real}} \prec M$, there is a possibly different commutative group operation $\oplus$ on $|K|$ which is defined over $\overline{\mathbb{Q}}^{\text{real}}$. 
Step 1: Semialgebraic case

Theorem ([BB11])

Let $G$ be a semialgebraic group of dimension $n$ over a real closed field $M$. Suppose $G$ is definably compact, definably connected, abelian. Then $G$ is definably homeomorphic to $T^n(M)$.

Proof.

- We can assume that $G$ is defined over $\mathbb{Q}^{real} \prec M$. Consider $G(\mathbb{R})$.
- $G(\mathbb{R})$ is a compact connected abelian Lie group with the t-topology.
- There is an analytic isomorphism $h : G(\mathbb{R}) \to T^n(\mathbb{R})$.
  - $h$ is definable in the o-minimal structure $\mathbb{R}_{an}$.
- By Shiota’s o-minimal Hauptvermutung there is a semialgebraic homeomorphism $f : G(\mathbb{R}) \to T^n(\mathbb{R})$.
  - Since $\mathbb{Q}^{real} \prec M$ we can take $f$ defined without parameters.
- The same formula gives a definable homeomorphism $f^M : G(M) \to T^n(M)$.

Step 2: Homotopy transfer

Recall that homotopy equivalence is much weaker than homeomorphism.

Example

The figure “8” is homotopy equivalent to $\mathbb{R}^2$ minus two points.

The homotopy category is more flexible than the topological category.

Fact

- $|K|$ and $|L|$ are homotopy equivalent if and only if they are semialgebraically homotopy equivalent.
- But two polyhedra $|K|$ and $|L|$ can be homeomorphic without being definably homeomorphic in any o-minimal expansion of $\mathbb{R}$. 
Step 2: Homotopy transfer

Let $M$ be an o-minimal expansion of a field, and let $X$ be a semialgebraic set defined over $\mathbb{Q}^{\text{real}} \subset M$. Look at $X(\mathbb{R})$ and $X(M)$.

Theorem (Homotopy transfer)

\begin{enumerate}
\item $\pi_1^{\text{def}}(X(M)) \cong \pi_1(X(\mathbb{R}))$ ([BO02]);
\item $H_\ast^{\text{def}}(X(M)) \cong H_\ast(X(\mathbb{R}))$ ([BO02]);
\item $H_\ast^{\text{def}}(X(M); \mathbb{Q}) \cong H_\ast(X(\mathbb{R}); \mathbb{Q})$ ([EO04] and duality);
\item $\pi_n^{\text{def}}(X(M)) \cong \pi_n(X(\mathbb{R}))$ ([BO09]).
\end{enumerate}

The superscript “def” refers to the relativization to the definable category and will be omitted in the sequel.

Failure of topological transfer (see [BO03, Shi10, BB11])

If $|K|(M)$ is a definable manifold, then $|K|(\mathbb{R})$ is a (PL) manifold. But if $|K|(\mathbb{R})$ is a topological manifold, $|K|(M)$ need not be a definable manifold.

Step 2: Homotopy transfer

Proposition

If $X$ is a definable set, $\pi_1(X)$ is finitely generated. Similarly $H_n(X)$ and $H^n(X)$ are finitely generated for all $n$.

Proof.

By homotopy transfer after triangulating $X$

However $\pi_n(X)$ may not be finitely generated for $n > 1$.

Example

Let $X = S^1 \wedge S^2$. Then $\pi_2(X) = \mathbb{Z}(\omega)$.

Indeed, for $n > 1$, $\pi_n(X) = \pi_n(\tilde{X})$ where $\tilde{X}$ is the universal cover of $X$. When $X = S^1 \wedge S^2$ the space $\tilde{X}$ is an infinite line with infinitely many copies $S^2$ attached to it, so clearly $\pi_2(\tilde{X})$ is not finitely generated as a group (however it is finitely generated as a $\mathbb{Z}[\pi_1(X)]$-module).
Step 3: Torsion

Theorem ([Str94a])
Let $G$ be a definable abelian group. Then the $k$-torsion subgroup $G[k]$ is finite.

Question ([PS99])
Suppose $G$ is definably compact. Is $G[k]$ non-empty?

Theorem ([EO04])
Let $G$ be a definably compact definably connected abelian group of dimension $n$. Then $G[k] \cong \mathbb{T}^n[k]$.

The proof depends on the study of $\pi_1(G)$. One shows that $\pi_1(G) \cong \pi_1(\mathbb{T}^n)$ and that $G[k] \cong \pi_1(G)/k\pi_1(G)$. So we need to study $\pi_1(G)$.

Step 3: Fundamental group

Theorem ([EO04])
Let $G$ be a definably compact definably connected abelian group of dimension $n$. Then $\pi_1(G) \cong \mathbb{Z}^n$.

Proof.
- Since $\text{dom}(G)$ is a definable set, $\pi_1(G)$ is finitely generated.
- Since $G$ is abelian, the map $p_k : G \to G$, $x \mapsto kx$, is a homomorphisms.
- It is also a definable covering map, so it induces an injective homomorphism $p_k^* : \pi_1(G) \to \pi_1(G)$, given by $[\gamma] \mapsto k[\gamma]$.
- Since this holds for every $k$, $\pi_1(G)$ is torsion free.
- Being also abelian and finitely generated, $\pi_1(G) \cong \mathbb{Z}^s$ for some $s$.
- The proof of $s = n$ uses the cup product in cohomology plus the theory of Hopf-spaces (proof omitted).
Step 4: Higher homotopy groups
We have seen that if $X$ is a definable set $\pi_m(X)$ may not be finitely generated. However:

**Lemma ([BMO10])**

*If $G$ is a definably connected definable group in $M$, then $\pi_m(G)$ is finitely generated for all $m$.***

**Proof.**

1. If $G$ is a real Lie group, then the fundamental group $\pi_1(G)$ acts trivially on $\pi_m(G)$ for $m > 1$, namely $G$ is a simple space.
2. (Serre 1953) If $X$ is a simple space and $H_m(X)$ is finitely generated for all $m$, then $\pi_m(X)$ is finitely generated for all $m$.
3. Being a definable group, our group $G$ is a simple space in the definable category.
4. Conclude by a suitable homotopy transfer.

Step 4: Higher homotopy groups

In the abelian case we get:

**Corollary ([BMO10])**

*Let $G$ be a definably connected definable abelian group in $M$. Then $\pi_m(G) = 0$ for all $m > 1$.***

**Proof.**

1. The morphism $p_k : G \to G, x \mapsto kx$, is a covering map, so it induces an injective endomorphism of $\pi_m(G)$ given by multiplication by $k$.
2. Since $m > 1$ this is actually an automorphism of $\pi_m(G)$ [BO09].
3. Since this holds for all $k$, we deduce that $\pi_m(G)$ is divisible.
4. Since it is also abelian and finitely generated, it must be zero.

In particular this gives a proof that $\pi_m(\mathbb{T}^n) = 0$ for $m > 0$ without factoring $\mathbb{T}^n$ into one-dimensional subgroups.
Step 4: Higher homotopy groups

Theorem ([BMO10])

Let $G$ be a definably connected definably compact definable abelian group in $M$. Then $G$ is definably homotopy equivalent to $\mathbb{T}^n(M)$.

Proof.
- By [EO04], $\pi_1(G) \cong \mathbb{Z}^n$.
- Consider the map $f : \mathbb{T}^n \to G$ sending $(t_1, \ldots, t_n) \in [0, 1)^n$ to $\gamma_1(t_1) + \ldots + \gamma_n(t_n)$ where $[\gamma_1], \ldots, [\gamma_n]$ are free generators of $\pi_1(G)$.
- Then clearly $f_* : \pi_1(\mathbb{T}^n) \cong \pi_1(G)$.
- Since $\pi_m(G) = 0$ for $m > 1$, $f$ induces an isomorphism on all the $\pi_m$’s.
- By the o-minimal version of Whitehead’s theorem ([BO09]) $f$ is a definable homotopy equivalence.

Step 5: Homotopy tori

We have seen that our group $G$ is definably homotopy equivalent to $\mathbb{T}^n(M)$ and we want to prove that it is definably homeomorphic to it.

We could try to transfer “Borel’s conjecture” from $\mathbb{R}$ to $M$:

Borel’s conjecture

Let $X$ be a PL-manifold homotopy equivalent to $\mathbb{T}^n(\mathbb{R})$ (considered as a PL-manifold under a standard triangulation). Then $X$ is PL-homeomorphic to $\mathbb{T}^n(\mathbb{R})$.

... but unfortunately this conjecture is false (it turns out that $X$ is indeed homeomorphic to $\mathbb{T}^n(\mathbb{R})$ but the homomorphism is not necessarily PL).
Step 5: Homotopy tori

As a partial substitute for Borel’s conjecture we have:

**Theorem (Homotopy tori)**

Let $X$ be a (closed) $PL$-manifold of dimension $n \neq 4$ homotopy equivalent to $T^n(\mathbb{R})$. Then there is a finite $PL$-covering $f : T^n(\mathbb{R}) \to X$.

**References**

For $n \geq 5$, see [HW69]. The case $n = 3$ follows from results in [KS77] plus the positive solution of Poincaré’s conjecture. For $n \leq 2$ $X$ is already $PL$-homeomorphic to $T^n(\mathbb{R})$.

Step 5: Homotopy tori

By triangulating our group $G$ and transferring from $\mathbb{R}$ to $M$ the result on Homotopy tori we get:

**Lemma**

There is a semialgebraic (even $PL$) finite cover

$$f : T^n(M) \to G(M)$$

(as spaces, not as groups).

This is nice, but it would have been more useful to have a cover going the other way around.
Step 5: Homotopy tori

Lemma

Let $G$ be definably compact, definably connected, abelian, of dim $\neq 4$. Then there is a definable finite cover $h : G(M) \to \mathbb{T}^n(M)$.

Proof.

- Start with the finite cover $f : \mathbb{T}^n(M) \to G(M)$ (as spaces).
- Lift the group operation on $G$ to a definable group operation $*$ on $\mathbb{T}^n$. So $f$ becomes a cover of definable groups $f : H = (\mathbb{T}^n, *) \to G$.
- Since $\ker(f) < H$ is finite, $\ker(f) < H[m]$ for some $m$.
- So we get a group cover $G \cong H/\ker(f) \to H/H[m]$.
- $H$ is a definable abelian group, so by [Str94a] $H[m]$ is finite. Deduce that $H/H[m] \cong H$ (as $H$ is definably connected).
- So we get a definable cover $h : G(M) \to \mathbb{T}^n(M)$.

Step 5: Reduction to the semialgebraic case

Lemma

Let $G$ be definably compact, definably connected, abelian, of dim $\neq 4$. Then $G$ is definably homeomorphic (not isomorphic!), to a semialgebraic abelian group.

Proof.

- We can assume that $\text{dom}(G) = |K|$ (triangulation).
- We have a definable cover $h : G \to \mathbb{T}^n$ (as spaces).
- Since $\mathbb{T}^n$ is semialgebraic, there is a semialgebraic cover $h' : G' \to \mathbb{T}^n$ and a definable homeomorphism $\phi : G \approx G'$ with $h' \circ \phi = h$. (see [EJP10])
- By the o-minimal Hauptvermutung there is a semialgebraic homeomorphism $\psi : G \approx G'$ and therefore $h' \circ \psi : G \to \mathbb{T}^n$ is a semialgebraic cover.
- $\mathbb{T}^n$ admits a semialgebraic commutative group operation which can be lifted to $G$ via the cover.
Conclusions

By a reduction to the semialgebraic case we have proved that a definably connected definably compact definable abelian group $G$ with $\dim(G) \neq 4$ is definably homeomorphic to $\mathbb{T}^n(M)$.

When $\dim(G) = 4$ the same proof gives the following weaker result:

**Corollary**

Suppose $\dom(G) = |K|(M)$. Then $|K|\mathbb{R}$ is homeomorphic to $\mathbb{T}^n\mathbb{R}$.

**Proof.**

By results in [FQ90] a PL-manifold homotopy equivalent to $\mathbb{T}^n\mathbb{R}$ is homeomorphic to $\mathbb{T}^n\mathbb{R}$. However in principle the homeomorphism may be wild (not PL), so we cannot deduce $|K|(M)$ is definably homeomorphic to $\mathbb{T}^n(M)$.

Conclusions

A possible approach to deal with the case $\dim(G) = 4$ is to replace $G$ with $G \times \mathbb{T}^1$ (a group of $\dim = 5$). But ...

**Question**

Let $X$ be a semialgebraic set and suppose that $X(\mathbb{R}) \times \mathbb{T}^1(\mathbb{R})$ is semialgebraically homeomorphic to $\mathbb{T}^5(\mathbb{R})$. Is $X(\mathbb{R})$ semialgebraically homeomorphic to $\mathbb{T}^4(\mathbb{R})$?

The answer is yes if $\mathbb{T}^1$ is replaced by $\mathbb{R}$ (Shiota). With $\mathbb{T}^1$ there could be problems.
Conclusions

Another possibility to deal with the dim = 4 case is to use the work of many authors on the “infinitesimal subgroup” $G^{00}$ of $G$ (see [Pil04, BOPP05]):

**Theorem**

1. If $G$ is a definable group in $M$ and $M^* \succ M$ is saturated, then $G = G(M^*)/G^{00}(M^*)$ is a compact real Lie group [BOPP05].
2. When $G$ is definably compact, $G^{00}$ is torsion free and $\dim_M(G) = \dim_{\mathbb{R}}(G)$ [HPP07].

When $G$ is definably compact, definably connected and abelian, it follows that $G \cong \mathbb{T}^n(\mathbb{R})$. We could try to transfer results from $G$ to $G$ and deduce that $G$ is definably homeomorphic to $\mathbb{T}^n(M)$.

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**Conclusions**

**Theorem (Transfer from $G$ to $G$)**

Let $G$ be definably compact and let $p : G \to \bar{G} = G(M^*)/G^{00}(M^*)$ be the projection.

1. $\dim_M(G) = \dim_{\mathbb{R}}(\bar{G})$ [HPP07].
2. The image under $p$ of a nowhere dense definable $X \subset G$, has measure zero in $\bar{G}$ [HP09].
3. If $U \subset G$ is open and $V \subset G$ is the preimage of $U$, then $\pi_1(U) \cong \pi_1^{\text{def}}(V)$ [BM11].
4. $\bar{G}$ determines the definable homotopy type of $G$ [Bar09, BM11].

Taking $G$ abelian and $U = \bar{G}$ in (3) we obtain $\pi_1(G) \cong \pi_1^{\text{def}}(\mathbb{G}) \cong \mathbb{Z}^n$. However (3) uses (2), which in turn uses $\pi_1(G) \cong \mathbb{Z}^n$.

**Conjecture**

If $G$ is triangulated by $|K|(M)$, then $G$ can be triangulated by $|K|(\mathbb{R})$. 
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