This is joint work with Tobias Kaiser and Jean-Philippe Rolin.
Let $\xi : S^2 \rightarrow TS^2$ be a real analytic vector field.

### Definition

A cycle $C$ of $\xi$ is a **limit cycle** if $C$ is contained in the (topological) closure of some non-compact trajectory of $\xi$.

### Dulac’s Problem

$\xi$ has finitely many limit cycles.

Ecalle and Il’yashenko independently found proofs of Dulac’s problem in the early 1990s, completing Dulac’s original strategy.
Assume $\xi$ has infinitely many limit cycles. Then they must pile up towards a nonempty, compact subset $\Gamma$ of $S^2$.

**Definition**

$\Gamma$ is called a **limit periodic set** of $\xi$.

Dulac showed that $\Gamma$ must be a **polycycle**, that is, a closed curve composed of finitely many singular points of $\xi$ connected by trajectories.

It remains to show that the **Poincaré first-return map** near $\Gamma$ has finitely many isolated fixed points.
If $\xi$ is polynomial, then the number of its limit cycles has a finite upper bound that depends only on the degree of $\xi$.

To approach H16, one has to study not just the individual $\xi$, but the whole family $\xi_\nu$ of polynomial vector fields of a given degree. This leads to complications:

1. More complicated limit periodic sets.
2. Study families of Poincaré first-return maps.

H16 has a vaguely model-theoretic flavor, but no model-theoretic framework has been found to exploit this.
Definition

Γ ⊆ S^2 is a **limit periodic set** of the family ξ_ν if there are limit cycles C_i of ξ_ν_i such that ν_i → ν and (C_i) converges in the sense of Hausdorff to Γ.

Roussarie’s conjecture

Let Γ ⊆ S^2 be a limit periodic set of ξ_ν. Then there are n ∈ ℕ and open neighbourhoods U of Γ in S^2 and V of ν in the parameter space such that ξ_µ has at most n limit cycles contained in U, for all µ ∈ V.

A topological compactness argument shows that Roussarie’s conjecture implies H16.
Our hope

...is to prove Roussarie’s conjecture when $\Gamma$ is a hyperbolic polycycle.

**Definition**

- A singularity $p$ of $\xi$ is **hyperbolic** if the linear part of $\xi$ at $p$ has two nonzero real eigenvalues of opposite signs.
- A polycycle of $\xi$ is **hyperbolic** if each of its singularities is hyperbolic.

Naive approach: let $\Gamma$ be a hyperbolic polycycle of $\xi_0$ and $P_\nu$ be the family of Poincaré first-return maps of $\xi_\nu$ near $\Gamma$, for $\nu$ near 0.

“Conjecture”

The expansion $\mathbb{R}^{P_\nu}$ of the real field by $P_\nu$ is o-minimal.
We assume 0 is a **hyperbolic** singularity of \( \xi \).

Let \( \gamma^- \) and \( \gamma^+ \) be two adjacent separating trajectories with limit point 0; we assume \( \gamma^- \) is incoming and \( \gamma^+ \) is outgoing.

We fix two segments \( \Lambda^- \) and \( \Lambda^+ \) transverse to \( \xi \) and equipped with analytic charts \( x \) and \( y \).

For some sufficiently small \( \epsilon > 0 \), we denote by \( g : (0, \epsilon) \rightarrow (0, \epsilon) \) the **transition map** of \( \xi \) from \( \Lambda^- \) to \( \Lambda^+ \).

**Strategy**

Prove that there exists an o-minimal expansion \( \mathcal{R} \) of \( \mathbb{R}_{an} \) in which all such transition maps are definable.
Theorem (Dulac and Ilyashenko)

Let $g$ be a transition map of $\xi$ near $0$. Then there is a series
\[ \hat{g} = p_0 X^{\nu_0} + \sum_{j=1}^{\infty} p_j(\log X)X^{\nu_j} \]
such that
\[ g(e^z) \text{ extends analytically to a quadratic domain} \]
\[ W = \left\{ z \in \mathbb{C} : \text{Re} \ z < r - C \sqrt{|\text{Im} \ z|} \right\} \text{ with } r \in \mathbb{R}, C > 0, \]
such that for every $n \geq 1$,
\[ g(e^z) - p_0 e^{\nu_0 z} - \sum_{j=1}^{n} p_j(z) e^{\nu_j z} = o \left( e^{\nu_n \text{Re} \ z} \right) \]
as $\text{Re} \ z \to -\infty$ in $W$. 

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O-minimality and Hilbert's 16th problem
We call $g \in D_\log$ satisfying (*) an **Ilyashenko function**, and we let $\mathcal{I}_\log$ be the set of all such germs.

**Theorem (Ilyashenko)**

1. If $g \in \mathcal{I}_\log$ is such that $\hat{g} = X$, then $g = x$ (*quasi-analyticity*).
2. $\mathcal{I}_\log$ is closed under composition.

**Corollary**

Let $\Gamma$ be a polycycle of $\xi$ such that every vertex of $\Gamma$ is a hyperbolic singularity of $\xi$. Then $\xi$ has finitely many limit cycles near $\Gamma$. 

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We let $Q$ be the subset of $I_{\log}$ consisting of all germs that do not contain log terms in their asymptotic expansions.

**Theorem (1)**

*There is a model-complete and o-minimal expansion $\mathbb{R}_Q$ of $\mathbb{R}_{an}$ in which every germ in $Q$ are definable. In particular, every transition map near a non-resonant hyperbolic singularity of $\xi$ is definable in $\mathbb{R}_Q$.*

What does this do for parametric families $\xi_\nu$?

**Proposition (2)**

*Assume $\xi_\nu$ is an analytic unfolding of $\xi_0$ such that each $\xi_\nu$ has only non-resonant hyperbolic singularities. Then the family of transition maps near each singularity of $\xi_\nu$ is definable in $\mathbb{R}_Q$.***
Next goals

We are now trying to extend

- Theorem (1) to all transition maps near *elementary* singular points of a single vector field $\xi$;
- Proposition (2) to an analytic family $\xi_\nu$ of vector fields with only hyperbolic singularities.

For both these extensions, the main difficulty lies in

- defining corresponding Ilyashenko functions in several variables;
- understanding blowings-up for these functions, in order to obtain a normalization algorithm.