Definable semi-germs.

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1. The ring $\mathcal{F}^{(n)}$

- $^*\mathbb{R}$ denotes an extension of $\mathbb{R}$ suitable for Non-Standard Analysis (NSA).

- ~~For~~ definiteness, $^*\mathbb{R} = \mathbb{R}^I/_\mathcal{U}$: a suitably saturated ultrapower of $\mathbb{R}$.
- Also, $^*\mathbb{C} := \mathbb{C}^I/_\mathcal{U}$ $(= {}^*\mathbb{R} \times {}^*\mathbb{R}$
$= {}^*\mathbb{R}[i])$.
- Indeed .....

... all functions $f : \mathbb{R}^n \to \mathbb{R}^m$, and all sets $S \subseteq \mathbb{R}^n$ have canonical extensions, denoted ${}^*f$, ${}^*S$ to ${}^*\mathbb{R}$.

- Łoś Theorem states that ${}^*\mathbb{R}$ is an elementary extension of $\mathbb{R}$ for any 1st order structure on $\mathbb{R}$.

- Let us fix an o-minimal structure on $\mathbb{R}$ expanding the field structure.
- Then, in particular, $\mathbb{R} \preccurlyeq {}^*\mathbb{R}$ for this structure, and ${}^*\mathbb{R}$ is o-minimal.

- "Definable" means definable in $^*\mathbb{R}$ (or in $\mathbb{R}$) <u>with</u> parameters, unless otherwise stated. This is also applied in $^*\mathbb{C}$ (and $\mathbb{C}$) via the identification $^*\mathbb{C} = {}^*\mathbb{R} \times {}^*\mathbb{R}$ (or $\mathbb{C} = \mathbb{R} \times \mathbb{R}$).
- It is convenient to assume that every $r \in \mathbb{R}$ is a constant of our language.

<u>Some notation</u>

For $\bar{r} \in \mathbb{R}_+^n$, $\Delta^{(n)}(\bar{r}) = \{\bar{z} \in \mathbb{C}^n : |z_i| < r_i,\ i = 1, \dots, n\}$ (where $\bar{r} = \langle r_1, \dots, r_n \rangle$).

So $^*\Delta^{(n)}(\bar{r})$ denotes its extension to $^*\mathbb{C}$.

Definition

(1) $\mathcal{F}^{(n)}(\bar{r}) := \{ f : {}^*\Delta^{(n)}(\bar{r}) \longrightarrow {}^*\mathbb{C} \mid f \text{ is holomorphic and definable} \}$.

(2) Clearly $\bar{r} < \bar{s}$ implies $\mathcal{F}^{(n)}(\bar{s}) \subseteq \mathcal{F}^{(n)}(\bar{r})$ and $\mathcal{F}^{(n)}$ denotes the direct limit of the directed set $\{ \mathcal{F}^{(n)}(\bar{r}) : \bar{r} \in \mathbb{R}^n_+ \}$

NOT ${}^*\mathbb{R}^n_+$

Then $\mathcal{F}^{(n)}$ is called the *differential ring* of *definable*, *holomorphic semi-germs* (in $\langle {}^*\mathbb{C}^n, \bar{0} \rangle$).

- Let $\mu :=$ set of infinitesimals of ${}^*\mathbb{C}$.
$= \{ \bar{z} \in \mathbb{C}^* : \forall r \in \mathbb{R}_+ , |z_i| < r \}$

Then each $f \in \mathcal{F}^n$ may be regarded as a function $f : \mu^n \to {}^*\mathbb{C}$ (having the property that it extends to a definable, holomorphic $f : {}^*\Delta^{(n)}(\bar{r}) \to {}^*\mathbb{C}$, for some $\bar{r} \in \mathbb{R}_+^n$).

- For any $\bar{\alpha} \in \mu^n$, we have a homomorphism

$$H_{\bar{\alpha}} : \mathcal{F}^{(n)} \longrightarrow {}^*\mathbb{C}[[z_1, .., z_n]] : f \longmapsto \sum_{\sigma \in \mathbb{N}^n} \frac{f^{(\sigma)}(\bar{\alpha})}{\bar{\alpha}!} \bar{z}^{\sigma}$$

<u>Proposition</u> (Peterzil - Starchenko)

This homomorphism $H_{\bar{\alpha}}$ is injective.

[Remark: Does not need poly-boundedness.]

- Situation for ${}^*\mathbb{R}^n$ is obscure.

I shall show here the following:

Theorem

Assume that the given o-minimal structure is polynomially bounded. Then $\mathcal{F}^{(m)}$ is Noetherian.

Remarks

1) $\mathcal{F}^{(m)}$ is not (algebraically) flat in $^*\mathbb{C}[[z_1,..,z_m]]$, e.g. if $\alpha \in \mu\setminus\{0\}$, then $z_1-\alpha$ is invertible in $^*\mathbb{C}[[z_1]]$ (via H_0), but not in $\mathcal{F}^{(m)}$.

2) I don't know if the theorem holds without poly-boundedness.

Motivation .....

2. Zilber's Conjecture

$\mathbb{C}_{exp}$ $(:= \langle \mathbb{C}; +, \cdot, \exp \rangle)$ is <u>quasi-minimal</u>, i.e. every $\mathbb{C}_{exp}$-definable subset of $\mathbb{C}$ is either countable or co-countable.

· This clearly involves studying sets of the form

$$V = \{\bar{z} \in \mathbb{C}^n : f_1(\bar{z}) = \cdots = f_m(\bar{z}) = 0\}$$

where $f_j(\bar{z}) \in \mathbb{C}[z_1, .., z_n, e^{z_1}, .., e^{z_n}]$.

<u>Crucial Case</u>: $m = n-1$ and V is non-singular w.r.t. $\bar{z}'$ $(= z_2, \ldots, z_n)$.

<u>Crucial Question</u> ....

What is the nature of the set of asymptotes of V, i.e. of the set of those $z_1 \in \mathbb{C}$ for which $\exists\, w_i \to z_1$ and $\langle w_i, \bar{z}_i' \rangle \in V$ with $\| \langle w_i, \bar{z}_i' \rangle \| \to \infty$ as $i \to \infty$.

Idea: Consider ${}^*V \subseteq {}^*\mathbb{C}^n$ and pick $i_0 \in {}^*\mathbb{N} \setminus \mathbb{N}$. Let $\bar{\alpha} = \langle w_{i_0}, \bar{z}_{i_0}' \rangle$. Then $\bar{\alpha} \in {}^*V$ and

$$ {}^*V \cap (\bar{\alpha} + {}^*\Delta^{(n)}(T)) $$

is definable (for the o-minimal, poly-bdd. structure ${}^*\mathbb{R} \succcurlyeq \langle \mathbb{R}; +, \cdot; \exp\restriction[0,1], \sin\restriction[0,1], \{r\}_{r \in \mathbb{R}} \rangle$) with parameters from ${}^*\mathbb{R}$ (actually ${}^*\mathbb{C}$).

3. Robinson's fundamental theorem of nonstandard complex analysis.

Let $\bar{r} \in \mathbb{R}_+^n$, $f \in \mathcal{F}^{(n)}(\bar{r})$ and assume that $\exists R \in \mathbb{R}_+$ such that $|f(\bar{z})| < R$ for all $\bar{z} \in {}^*\Delta^{(n)}(\bar{r})$.

Then we may define a function $\tilde{f} : \Delta^{(n)}(\bar{r}) \longrightarrow \mathbb{C}$ by $\tilde{f}(\bar{z}) = st.pt.(f(\bar{z}))$.

Then $\tilde{f}$ is holomorphic and $\frac{\partial \tilde{f}}{\partial z_i} = \widetilde{\frac{\partial f}{\partial z_i}}$ on $\Delta^{(n)}(\bar{r})$.

Remarks

(1) False in real case, e.g. consider

$f : [-1,1] \longrightarrow [0,1] : x \longmapsto \dfrac{\varepsilon}{x^2+\varepsilon}$ (ε a positive inf^{mal}). $\tilde{f}$ not even continuous.

(2) The Marker-Steinhorn theorem implies that $\tilde{f}$ is also definable (in the o-minimal structure on $\mathbb{R}$), and hence ${}^*\tilde{f} \in \mathcal{F}^{(n)}(\bar{r})$, and notice that ${}^*\tilde{f}$ is definable without parameters. More on this later.

Proof of Robinson's Theorem.

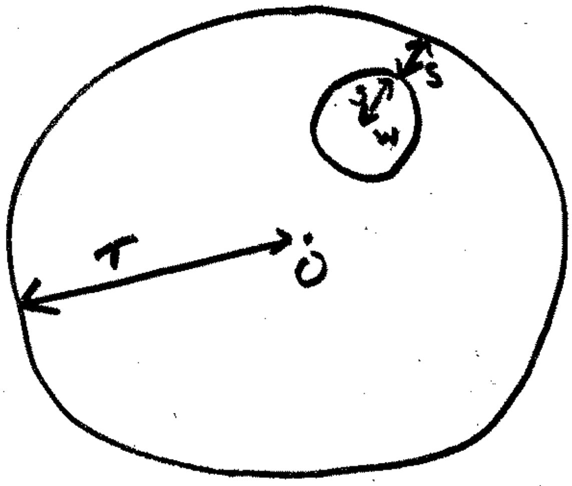
- One easily reduces to case $n = 1$.
- Let $f : {}^*\Delta^{(1)}(r) \to {}^*\Delta^{(1)}(R)$, $f \in \mathcal{F}^{(1)}(r)$.

  Let $w \in \Delta^{(1)}(r)$ and set $s = \frac{1}{2}(r - |w|)$.

  Then f is $\frac{2R}{s}$-Lipschitz on ${}^*\Delta^{(1)}_w(s)$....

.... just apply the Maximum Modulus theorem (in ${}^*\mathbb{C}$ – this is NSA) to

$$F(z) := \frac{f(z) - f(w)}{z - w} \quad \text{for} \quad |z-w| < \Delta \ :-$$



Since $\frac{2R}{\Delta} \in \mathbb{R}_+$, this easily gives continuity, and same Lipschitz property for $\tilde{f}$. For analyticity, can use Morera's theorem, or same argument applied to

$$G(z) := \frac{f(z) - f(w) - (z-w) f'(w)}{(z-w)^2} \qquad \square$$

4. Using the Marker-Steinhorn Theorem.

- We have ${}^*\mathbb{C} \le \mathcal{F}^{(n)}$ (as constant functions).
- If I is an ideal of $\mathcal{F}^{(n)}$ then $I \cap {}^*\mathbb{C} = \{0\}$
- I prefer to work with the rings

$$\mathcal{A}^{(n)}(\bar{r}) := \{ f \in \mathcal{F}^{(n)} : \exists R \in \mathbb{R},\ |f(\bar{z})| < R,\ \forall \bar{z} \in {}^*\Delta^{(n)}(\bar{r}) \}.$$

and the corresponding direct limit, denoted $\mathcal{A}^{(n)}$.

- Sufficient to show $I \cap \mathcal{A}^{(n)}$ is a fin. gen. ideal of $\mathcal{A}^{(n)}$, since $\forall f \in \mathcal{F}^{(n)}$ $\exists b \in {}^*\mathbb{C}$ s.th. $b \cdot f \in \mathcal{A}^{(n)}$.
- For $n = 0$, $\mathcal{A}^{(0)} = \mathrm{Fin}({}^*\mathbb{C})$ and $I \cap \mathcal{A}^{(0)} = (0)$.

Let $n > 0$.

Let $\mathcal{A}_0^{(n)}(\bar{r})$ be the (differential) subring of $\mathcal{A}^{(n)}(\bar{r})$ consisting of only the parameter-free definable $f \in \mathcal{A}^{(n)}(\bar{r})$.

Let $\mathcal{A}_0^{(n)}$ be the corresponding direct limit of the $\mathcal{A}_0^{(n)}(\bar{r})$'s.

By Robinson's Theorem and the Marker-Steinhorn Theorem we have a differential ring homomorphism

$$\phi_n : \mathcal{A}^{(n)} \twoheadrightarrow \mathcal{A}_0^{(n)} : f \longmapsto {}^*\tilde{f}$$

(which splits the inclusion $\mathcal{A}_0^{(n)} \subseteq \mathcal{A}^{(n)}$).

NB: For $f \in \mathcal{A}^{(n)}(\bar{r})$, $|f(\bar{z}) - {}^*\tilde{f}(\bar{z})| \in \mu$ $\forall \bar{z} \in {}^*\Delta^{(n)}(\bar{r})$.

So $f \ (\in \mathcal{A}^{(n)})$ lies in $\ker(\phi_n)$ iff $\exists \bar{r} \in \mathbb{R}_+^n$ s.th. $f \in \mathcal{A}^{(n)}(\bar{r})$ and $|f(\bar{z})| \in \mu \quad \forall \bar{z} \in {}^*\Delta^{(n)}(\bar{r})$.

We now obtain (by transfer):

5. <u>The WPT for semi-germs</u>

Suppose $f \in \mathcal{A}^{(n)} \setminus \ker(\phi_n)$ and $f(\bar{0}) = 0$. Then after a linear, homogeneous change of coordinates, $\exists \ d \geq 1$, there exist $\bar{r} \in \mathbb{R}_+^n$ and a representative $f \in \mathcal{A}^{(n)}(\bar{r})$ with a ~~representation~~

$$f(\bar{x}', x_n) = u(\bar{x}', x_n)\left(x_n^d + a_1(\bar{x}')\cdot x_n^{d-1} + \dots\right)$$

with $u(\bar{0}', 0) \notin \mu$, $u \in \mathcal{A}^{(n)}(\bar{r})$, a_i's $\in \mathcal{A}_1^{(n-1)}(\bar{r}')$

This only uses o-minimality. We can now complete the proof that I is finitely generated in the usual way, <u>except</u> for the hypothesis in WPT that $f \notin \ker(\phi_n)$. (*)

- May assume $\sup_{\bar{z} \in {}^*\Delta^{(n)}(\bar{r})} |f(\bar{z})| = 1$.

However, this is not enough to guarantee that $f \notin \ker(\phi_n)$.

E.g. for $\varepsilon > 0$, $\varepsilon \in \mu$, consider $\frac{\varepsilon}{z - (1+\varepsilon)} = g(z)$

Then $\sup_{z \in {}^*\Delta^{(1)}(1)} |g(z)| = 1$, $\sup_{z \in {}^*\Delta^{(1)}(1/2)} |g(z)| < 4\varepsilon$

So $g \in \ker(\phi_1)$.

(*) [Crucial point: $\mathcal{A}^{(n)}$ is <u>not</u> Noetherian since $\ker(\phi_n)$ is a non-finitely generated ideal of $\mathcal{A}^{(n)}$ (but not of $\mathcal{F}^{(n)}$). "..the usual way" requires that $u(\bar{x}', x_n)$ be invertible in $\mathcal{A}^{(n)}$.]

Theorem

Assume that our o-minimal structure is polynomially bounded. Let $f \in \mathcal{A}^{(n)}(\bar{r})$. Then there exists $\bar{s} \in \mathbb{R}_+^n$ with $\bar{s} < \bar{r}$, and $c \in {}^*\mathbb{C}$ such that

$$\sup_{\bar{z} \in {}^*\Delta^{(n)}(\bar{s})} |cf(\bar{z})| = \sup_{\bar{z} \in {}^*\Delta^{(n)}(\bar{s})} |{}^*c\hat{f}(\bar{z})| = 1$$

and hence $c \cdot f \in \mathcal{A}^{(n)} \setminus \ker(\phi_n)$.

.... and of course $c \cdot f \in I \Leftrightarrow f \in I$.